On Triangular and Trapezoidal Numbers

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Abstract –*The nth triangular number, denoted by T (n), is defined as the sum of the first consecutive positive integers. It can be represented in the form of a triangular grid of points. As a result, this study discusses the triangular numbers and its properties. To distinguish whether a positive integer* N *is a triangular number or not, it has to comply with* 8N + 1 *which is a perfect square; characteristics of odd and even triangular numbers; sum of two consecutive triangular numbers with same parity i.e., the formula for* $T(4c-1)+T(4c)=(4c)^2$ *where* $c \geq 1$ *and* $T(4c+1)+T(4c+2)=(4c+2)^2$ *where* $c \geq 0$*; and introduces the new figurative numbers through deriving the formula of triangular number, Trapezoidal Number- an integer of* $T(n)-1$*, denoted by* $T^*(n)$*and Diamond Number- an integer of* $2T(n)-n$*, denoted by* D (n)*.*

Keywords: *Triangular numbers, Odd and Even Triangular numbers, Parity of triangular numbers, Trapezoidal number, Diamond number*

INTRODUCTION

Triangular numbers, in fact, is a family of numbers. They are figurate numbers that can be able to characterize as a standard and distinct geometric pattern. For many centuries, the triangular numbers have equally fascinated and puzzled scholars around the world. The triangular numbers are so named since they correspond to the number of dots or other object that build up the shape of an equilateral triangle [1].

Bounding to the fore of present day, triangular numbers have developed as an essential function of education.

This paper is anchored on a research paper "On the Generalized Formula for the Triangular Number of the Sum and Product of Natural Numbers" which was presented by Jonjie M. Catibog in March 2012 at De La Salle Lipa. This study simplifies the formulas for the triangular number of the sum and product of *r* natural numbers. For any natural numbers $n_1, n_2, n_3, \ldots, n_r, r \geq 2$, the generalized formula for the triangular number of the sum of *r* natural number is given by

$$T\left(\sum_{i=1}^{r} n_i\right) = \sum_{i=1}^{r} T(n_i) + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} n_i n_j$$

The formula for the triangular number of the sum of *r* natural numbers is utilized to generalize the formula for the sum of an odd number of consecutive triangular numbers, *i.e.*

$$\sum_{i=1}^{2k+1} T(n_i) = \sum_{i=-k}^{k} T(n+i) = (2k+1)T(n) + \frac{k(k+1)(2k+1)}{6}$$

,and the formula for the sum of an even number of consecutive triangular number, *i.e.*

$$\sum_{i=1}^{2k} T(n_i) = \sum_{i=0}^{2k-1} T(n+1) = 2kT(n+k-1) + kn + \frac{k}{3}(k^2+3k-1)$$

. Hence, for any natural numbers $n_1, n_2, n_{3,\ldots}, n_r, r \geq 2$, the generalized formula for finding the triangular number of the product of *r* natural numbers is given by

$$T\left(\prod_{i=1}^{r} n_i\right) = \prod_{i=1}^{r} T(n_i) + \sum_{\substack{i=1 \\ i \geq i_2}}\prod_{\substack{k=i,i_2 \\ j \neq i,i_2}}^{r} T(n_k-1)T(n_j) +$$

$$\sum_{\substack{i=1 \\ i<i_2<i_3<i_4}}\prod_{\substack{k=i,i_2,i_3,i_4 \\ j\neq i,i_2,i_3,i_4}}^{r} T(n_k-1)T(n_j) + \ldots + \sum_{\substack{i=1 \\ i<i_2<\ldots i_{r-1}}}\prod_{\substack{k=i,i_2,\ldots i_{r-1}i_r \\ j\neq i,i_2,\ldots i_{r-1},i_r}}^{r} T(n_k-1)T(n_j).$$

Applications of triangular numbers, such as the Twelve Days of Christmas and counting the number of edges and diagonals of polygons, are presented in this exposition. This study was tied in the article "A generalization of the formula for the triangular number of the sum and product of natural numbers"

which was published in International Journal of Mathematical Education in Science and Technology in October 2008. This was authored by Dr. Muniru A. Asiru [2] of the Department of Mathematics and Statistics of the Federal Polytechnic at Niger State, Nigeria.

This paper examined a special type of figurate number, the triangular number. In general, a triangular number is the number of dots that can be arranged to form an equilateral triangle. One purpose of this study is to provide a comprehensive explanation of this type of number. Though these numbers are often called the sum of the first n consecutive positive integers throughout discussions, they are rarely considered as a particular type of number. This also aims to present the characteristics of odd and even triangular numbers and new approach in finding the formulas for the sum of two consecutive triangular numbers with same parity and introduce new figurative numbers.

This study aimed to present the triangular numbers. Specifically, this study provides clear understanding that intends to: (1) Discuss the triangular numbers and its properties (2) Present the characteristics of odd and even triangular numbers (3) Discuss the formulas for the sum of two consecutive triangular numbers with same parity (4) and to introduce new figurative numbers; trapezoidal numbers and diamond numbers.

RESULTS AND DISCUSSION
Triangular Numbers and Its Properties

This part provides a comprehensive explanation about triangular numbers and its properties: definition, theorems and examples.

Definition 3.1.1 A *triangular number T (n)* is a figurate number that can be represented in the form of a triangular grid of points

Remarks. The nth triangular number can also be defined as the sum of the first n consecutive natural numbers, *i.e.*

$$T(n) = \sum_{i=1}^{n} i = 1 + 2 + 3 + ... + n$$

The theorem below gives a necessary condition for the triangular numbers.

Theorem 3.1.1 For any natural number n, the nth triangular number is defined by

$$T(n) = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

Proof. Using the principle of mathematical induction on n, we show

$$T(n) = \frac{n(n+1)}{2}.$$

i.) For $n=1$, we obtain $1 = \frac{1(2)}{2} = 1$.

Hence, $T(1)$ is true.

ii.) Assume that for $n=1, 2, 3, …, k$

$$T(k) = 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$$

is true. We want to show that

$$T(k+1) = 1 + 2 + 3 + ... + (k+1) = \frac{(k+1)(k+2)}{2}$$

is also true.

Now,

$$T(k+1) = 1 + 2 + 3 + ... + k + (k+1)$$
$$= (1 + 2 + 3 + ... + k) + (k+1)$$
$$= \frac{k(k+1)}{2} + (k+1)$$

By simplifying, we get

$$T(k+1) = \frac{k^2 + k + 2(k+1)}{2}$$
$$= \frac{k^2 + k + 2k + 2}{2}$$
$$= \frac{k^2 + 3k + 2}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

(By Hypothesis of Induction)

Thus, $T(k+1)$ holds. By Principle of Mathematical induction, the above formula is true for all $n≥1$.

Corollary 3.1.1 For each natural number $n ≥ 1$, the integer $\binom{n+1}{2}$ is the nth triangular number.

Proof. By definition, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Thus,

$$\binom{n+1}{2} = \frac{(n+1)!}{2![(n+1)-2]!}$$
$$= \frac{(n+1)!}{2(n-1)!}$$
$$= \frac{(n+1)(n)(n-1)!}{2(n-1)!}$$

$$\binom{n+1}{2} = \frac{n(n+1)}{2} = T(n) \qquad ∎$$

The theorem below gives a necessary condition for the sum of two consecutive triangular numbers.

Theorem 3.1.2 The sum of two consecutive triangular numbers is always a perfect square.
Proof. Let $n \in Z^+$,

$$T(n) + T(n+1) = \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2}$$

$$= \frac{n^2+n}{2} + \frac{n^2+n+2n+2}{2}$$

$$= \frac{n^2+n+n^2+3n+2}{2}$$

$$= \frac{2n^2+4n+2}{2}$$

$$= \frac{2(n^2+2n+1)}{2}$$

$$= n^2+2n+1$$

$$= (n+1)^2$$

Remark. The sum of two consecutive triangular numbers $T(n)$ and $T(n+1)$ is the square of $(n+1)$.

The theorem below gives a necessary condition for $T(n)$ which is a perfect square.

Theorem 3.1.3 A natural number N is a triangular number if and only if 8N + 1 is a perfect square.
Proof. $(\Rightarrow)$ Suppose N is a triangular number. We show 8N+1 is a perfect square. Then there exist a positive integer *n* such that N = $T(n)$. By Theorem 3.1.1,

$$N = T(n) = \frac{n(n+1)}{2}$$

Multiplying both sides of the equation by 8, we have

$$8N = 8 \cdot \frac{n(n+1)}{2}$$

$$8N = 4n(n+1)$$

$$8N = 4n^2 + 4n$$

$$8N + 1 = 4n^2 + 4n + 1$$

$$8N + 1 = (2n+1)^2$$

Hence, $8N+1$ is a perfect square.
$(\Leftarrow)$ Assume $8N+1$ is a perfect square. We show N is a triangular number. Since 8N is even, it follows that 8N + 1 is odd.

Since 8N + 1 is odd and a perfect square, then there exist $k \in Z$ such that

$$8N + 1 = (2k+1)^2$$

$$= (2k+1)(2k+1)$$

$$= 4k^2 + 4k + 1$$

Subtracting 1 from both sides, we have

$$8N + 1 - 1 = 4k^2 + 4k + 1 - 1$$

$$8N = 4k^2 + 4k$$

Factoring out the greatest common divisor on the right side,

$$8N = 4k(k+1)$$

Dividing both sides by 8, we have

$$N = \frac{k(k+1)}{2}$$

Since $k \in Z^+$, N is a triangular number.

An immediate consequence of the above theorem is given below.
Corollary **3.1.2** Let *n* be a positive integer. Then,

$$8[T(n)] + 1 = (2n+1)^2.$$

Odd and Even Triangular Numbers
This section provides discussions about odd and even triangular numbers.

Definition 3.2.1 Let $n \in Z^+$. If $T(n)$ is even, then it is called *even triangular number*, denoted by $T_e(n)$.
Definition 3.2.2 Let $n \in Z^+$. If $T(n)$ is odd, then it is called *odd triangular number*, denoted by $T_o(n)$.

The theorem below gives a necessary condition for $T(n)$ to be even.
Theorem 3.2.1 Let $n \in Z^+$. If $4 \mid n$, then $T_e(n)$.
Proof. Suppose $4 \mid n$, then there exits an integer *c* such that $n = 4c$. It follows that

$$T(n) = T(4c)$$

$$T(4c) = \frac{4c(4c+1)}{2}$$

$$= \frac{16c^2 + 4c}{2}$$

$$= 4\left(\frac{4c^2 + c}{2}\right)$$

$$= 2(4c^2 + c)$$

Observe that $4c^2 + c$ is an integer. Hence, $T_e(n)$.

The theorem below gives a necessary condition for $T(n)$ to be odd.

Theorem 3.2.2 Let n is even integer. If $4 \mid n$, then $T_o(n)$.

Proof: Suppose $4 \mid n$, by Division of Algorithm, n is either $4c+1, 4c+2$ or $4c+3$, for some $c \in Z$. Since n is even, then $n = 4c+2$. It follows that

$$T(n) = T(4c+2)$$
$$T(4c+2) = \frac{(4c+2)(4c+2+1)}{2}$$
$$= \frac{(4c+2)(4c+3)}{2}$$
$$= \frac{16c^2 + 8c + 12c + 6}{2}$$
$$= \frac{16c^2 + 20c + 6}{2}$$
$$= 2\left(\frac{8c^2 + 10c + 3}{2}\right)$$
$$= 8c^2 + 10c + 3$$
$$= 8c^2 + 10c + 2 + 1$$
$$= 2(4c^2 + 5c + 1) + 1$$

Since $= 2(4c^2+5c+1)+1$ $(4c^2+5c+1)$ is an integer, say q. Hence, $T(n)$ is $2q+1$. Therefore $T(n)$ is odd.

Theorem 3.2.3 Let n be a positive odd integer. If $4 \mid n+1$, then $T_e(n)$.

Proof: Let $n \in Z^+$. If $4 \mid n+1$, then there exits an integer c such that $n+1 = 4c$. It follows that

$$T(n) = T(4c-1)$$
$$T(4c-1) = \frac{(4c-1)(4c-1+1)}{2}$$
$$= \frac{(4c-1)(4c)}{2}$$
$$= \frac{16c^2 - 4c}{2}$$
$$= 4\left(\frac{4c^2 - c}{2}\right)$$
$$= 2(4c^2 - c)$$

Since $4c^2 - c$ is an integer. Hence $T(n)$ is even.

An immediate consequence of the above theorem is given below.

Corollary **3.2.1** $T(4c-1)$ is even triangular number, for each integer $c \geq 1$.

Theorem 3.2.4 Let n be a positive odd integer. If $4 \nmid n+1$, then $T_o(n)$.

Proof: Suppose $4 \mid n+1$. By Division of Algorithm, $n + 1$ is exactly one of the form $4c+1$, $4c+2$ or $4c+3$. Since n is odd, then $n+1$ is even. Hence, $n+1 = 4c+2$. This implies that $n = 4c+1$. Now,

$$T(n) = T(4c+1)$$
$$T(4c+1) = \frac{(4c+1)(4c+1+1)}{2}$$
$$= \frac{(4c+1)(4c+2)}{2}$$
$$= \frac{16c^2 + 4c + 8c + 2}{2}$$
$$= \frac{16c^2 + 12c + 2}{2}$$
$$= 2\left(\frac{8c^2 + 6c + 1}{2}\right)$$
$$= 8c^2 + 6c + 1$$
$$= 2(4c^2 + 3c) + 1$$

where $(4c^2 + 3c)$ is an integer. Hence, $T(n)$ is odd.

Corollary **3.2.2** Let $n \in Z^+$. Then, $T(4n + 1)$ is an odd integer.

Sum of Two Consecutive Triangular Numbers with same Parity

The theorem below gives necessary condition for two consecutive even triangular numbers with same parity.

Theorem 3.3.1 For any two even triangular numbers, $T(4c-1)$ and $T(4c)$, then $T(4c-1) + T(4c) = (4c)^2$, where $c \geq 1$.

Proof. By Theorem 3.2.1 and Theorem 3.2.3, we have
$$T(4c-1) + T(4c) = \frac{(4c-1)(4c-1+1)}{2} + \frac{(4c)(4c+1)}{2}$$
$$= \frac{(4c-1)(4c)}{2} + \frac{(4c)(4c+1)}{2}$$
$$= \frac{16c^2 - 4c}{2} + \frac{16c^2 + 4c}{2}$$
$$= \frac{16c^2 + 16c^2 - 4c + 4c}{2}$$
$$= \frac{32c^2}{2}$$
$$= 16c^2$$
$$= (4c)^2$$

The theorem below gives necessary condition for two consecutive odd triangular numbers with same parity.

Theorem3.3.2 For any two odd triangular numbers, $T(4c+1)$ and $T(4c+2)$, then
$T(4c+1) + T(4c+2) = (4c^2+2)^2$, where $c \geq 0$.
Proof. By Theorem 3.2.2 and Theorem 3.2.4, we have

$$T(4c+1) + T(4c+2) = \frac{(4c+1)(4c+1+1)}{2} + \frac{(4c+2)(4c+2+1)}{2}$$

$$= \frac{(4c+1)(4c+2)}{2} + \frac{(4c+2)(4c+3)}{2}$$

$$= \frac{16c^2+8c+4c+2}{2} + \frac{16c^2+12c+8c+6}{2}$$

$$= \frac{16c^2+12c+2}{2} + \frac{16c^2+20c+6}{2}$$

$$= \frac{16c^2+16c^2+12c+20c+2+6}{2}$$

$$= \frac{32c^2+32c+8}{2}$$

$$= 16c^2+16c+4$$

$$= (4c+2)^2$$

New Figurative Numbers and Its Basic Properties
Trapezoidal Numbers
Definition. Let $T(n)$ a triangular number, for some integer $n \geq 2$. An integer of $T(n)-1$ is called *trapezoidal number*, denoted by $T*(n)$.

Diamond Numbers
Definition. Let $T(n)$ a triangular number. An integer of $2T(n)-n$ is called *diamond number*, denoted by $D(n)$.
Theorem 3.4.2.1. Let $n \geq 1$ be an integer. Then $D(n) = n^2$.
Proof: By definition,

$$D(n) = 2T(n) - n$$

$$= 2\left[\frac{n(n+1)}{2}\right] - n$$

$$= n(n+1) - n$$

$$= n^2 + n - n$$

$$= n^2 \qquad \blacksquare$$

CONCLUSION AND RECOMMENDATION
Triangular numbers are figurate numbers that can be able to characterize as a standard and distinct geometric pattern. So based on the results obtained,

the researchers drawn conclusions as per the purpose of the study. To distinguish whether a positive integer N is a triangular number or not, it has to comply with 8N + 1 which is a perfect square. And for any two consecutive triangular numbers, its sum is always a perfect square.

A further underlying principle of this research presents the characteristics of odd and even triangular numbers and based on the results, the researchers acquired discussions to characterize the odd and even triangular numbers and thus, any positive integer *n*, where 4 | *n* and 4 | *n* + 1 has been proved in distinguishing the even triangular number and 4 | *n* and 4 | *n* + 1 for odd triangular numbers. Through this study, the researchers scrutinized formulas for two consecutive triangular numbers with same parity. For any two even triangular numbers, and *T* (4*c*), then *T* (4*c* − 1) + *T* (4*c*) = (4*c*)^2 and for any two odd triangular numbers, *T* (4*c* + 1) and *T* (4*c* + 2), then *T* (4*c* + 1) + *T* (4*c* + 2) = (4*c* + 2) ^{2}.

Another further rationale of this research is to introduce new figurative numbers drawn from the formula of triangular number. In particular, any *T* (*n*) that is a triangular number, for some integer *n*≥2, an integer of *T* (*n*) − 1 is called trapezoidal number, denoted by *T** (*n*). Furthermore, for any triangular number *T* (*n*), where *n* ≥ 2, an integer of 2*T* (*n*) − *n* is called diamond number, denoted by *D* (*n*).

The researchers recommend this study "On Triangular and Trapezoidal Numbers" as an instructional reference material for instructors of mathematics, as well as available literature for future researches.

For the computer programming students, the researchers recommended the formula for trapezoidal and diamond numbers to be programmed.

It is also recommended to further analyze the interrelationships between triangular numbers and other figurate numbers since these numbers involve a lot of interesting patterns, both in algebraic and geometric aspects. And find the identities that show the connection between various figurate numbers.

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