On the Sum of Corresponding Factorials and Triangular Numbers: Some Preliminary Results

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Abstract – *A new sequence of natural numbers can be formed by adding corresponding factorials and triangular numbers. In this paper, such numbers were named factoriangular numbers. Mathematical experimentations on these numbers resulted to the establishment of some of its characteristics. These include the parity, compositeness, the number and sum of its positive divisors, abundancy and deficiency, Zeckendorf's decomposition, end digits, and digital roots of factoriangular numbers. Several theorems and corollaries were proven and some conjectures were also presented.*

Keywords – *factorial, factoriangular number, triangular number*

INTRODUCTION

Number theory is the study of the properties of integers and rational numbers beyond the usual manipulations of ordinary arithmetic. Because of its unquestioned historical importance, this theory has occupied a central position in the world of both ancient and contemporary mathematics. It has shown its irresistible appeal to mathematicians; one reason for this as Burton stated in [1], lies in the basic nature of its problems. Although many of the number theory problems are extremely difficult to solve and remain to be the most elusive unsolved problems in mathematics, they can be formulated in terms that are simple enough to arouse the interest and curiosity of even those without much mathematical training.

Exploring the characteristics of integers and patterns of integer sequences is one of the most interesting and frequently conducted studies in number theory. It is quite difficult now to count the number of studies on Fibonacci sequence, Lucas sequence, the Pell and associated Pell sequences, and other well-known sequences. Classical number patterns like the triangular numbers and other polygonal and figurate numbers have also been studied from the ancient up to the modern times. The multiplicative analog of the triangular number, the factorial, has also a special place in the literature being very useful not only in number theory but also in other mathematical disciplines like mathematical analysis and combinatorial theory.

Integer patterns or sequences can be described by algebraic formulas, recurrences and identities. For example, the nth triangular number (T_n), for $n \geq 1$, can be determined by the formula $T_n = n(n+1)/2$. Given a T_n, the next term in a sequence of triangular numbers can also be determined through the recurrence relation, $T_{n+1} = T_n + n + 1$. Some important identities on triangular numbers can be found in [2]-[4]. There is also an identity involving triangular number and factorial which is given by $(2n!) = 2^n \prod_{k=1}^{n} T_{2k-1}$ (see [5], [6]). Aside from this, there is a somewhat natural connection between factorials and triangular numbers. Factorial is defined for a positive integer n as $n! = 1 \cdot 2 \cdot 3 \cdots n$ and hence, the triangular number, which is defined for a positive integer n as $T_n = 1 + 2 + 3 + ... + n$, is regarded as the additive analog of factorial.

This natural similarity of the two numbers incites the present investigator to add corresponding numbers of the sequences of factorials and triangular numbers to form a new sequence of natural numbers, {2, 5, 12, 34, 135, 741, 5068, 40356, 362925,…}, which will be the subject of another experimentation and exploration. Curious enough, the author checked if such sequence is already included in Sloane's *The On-Line Encyclopedia of Integer Sequences* (OEIS) [7] and found that as sequence A101292. However, there is very little information about the sequence in OEIS, in particular, and in the literature, in general.

Hence, this study was conducted to explore and experiment on the natural numbers formed by adding corresponding factorials and triangular numbers and present some preliminary results on the parity, prime factors, number theoretic functions, abundancy and deficiency, positive divisors, Zeckendorf's decomposition, end digits, and digital roots of such numbers. For easy reference and recall, in this study, such a number is named *factoriangular number*, which is coined from the words factorial and triangular.

METHODS

More than in any part of mathematics, the methods of inquiry in number theory adhere to the scientific approach. In working on this study, the author relies to a great extent on trial and error, curiosity, intuition and ingenuity. Rigorous mathematical proofs are preceded by patient and time-consuming mathematical experimentation or experimental mathematics, which as defined in Nguyen [8], is the methodology of doing mathematics that includes the use of computations for gaining insight and intuition, discovering new patterns and relationships, using graphical displays to suggest underlying mathematical principles, testing and especially falsifying conjectures, exploring a possible result to see if it is worth a formal proof, suggesting approaches to formal proof, replacing hand derivations with computer-based derivations, and confirming analytically derived results. In the prime factorization of factoriangular numbers, the author used elliptic curve method [9].

RESULTS AND DISCUSSION

Corresponding factorials and triangular numbers are added here and the sums are named *factoriangular numbers*. The following notations are used: n for natural numbers, n! for factorial of a natural number, T_n for triangular number, and Ft_n for factoriangular number. The n!, T_n, and Ft_n, for $n \leq 20$, are given in Table 1.

A factoriangular number is defined as follows:
Definition. The nth factoriangular number is given by the formula $Ft_n = n! + T_n$, *where* $n! = 1 \cdot 2 \cdot 3 \cdots n$ *and* $T_n = 1 + 2 + 3 + ... + n = n(n+1)/2$.

Factoriangular number can also be defined as the sum of the first n natural numbers plus the factorial of n, that is $Ft_n = 1 + 2 + 3 + ... + n + n!$.

Table 1. The First 20 Factoriangular Numbers

N	n!	T_n	Ft_n
1	1	1	2
2	2	3	5
3	6	6	12
4	24	10	34
5	120	15	135
6	720	21	741
7	5040	28	5068
8	40320	36	40356
9	362880	45	362925
10	3628800	55	3628855
11	39916800	66	39916866
12	479001600	78	479001678
13	6227020800	91	6227020891
14	87178291200	105	87178291305
15	1307674368000	120	1307674368120
16	20922789888000	136	20922789888136
17	355687428096000	153	355687428096153
18	6402373705728000	171	6402373705728171
19	121645100408832000	190	121645100408832190
20	2432902008176640000	210	2432902008176640210

In characterization factoriangular numbers, their parities were first examined here. Notice from Table 1 that after the first and second, which is even and odd integer, respectively, factoriangular numbers are in alternating pairs of even and pairs of odd integers. This parity pattern is a result of a simple property of arithmetic: *the sum of two integers of the same parity is even and that of different parities is odd*. As defined earlier, factoriangular number is the sum of the analogous factorial and triangular number. The factorial of 1 is odd and for $n \geq 2$, the factorial of n is even, being 2 or multiple of 2. Triangular numbers, on the other hand, are in alternating pairs of odd and pairs of even integers. Adding these corresponding factorials and triangular numbers resulted to the series of factoriangular numbers with parity pattern mentioned earlier.

To further examine the parity of factoriangular numbers, let the natural number n be written in one of the forms 4k, 4k+1, 4k+2 or 4k+3, for integer $k \geq 0$, as shown in Table 2.

Notice that Ft_n is even if $n = 1$ or if n is of the form 4k, for integer $k \geq 1$, or of the form 4k+3, for integer $k \geq 0$; and Ft_n is odd if n is of the form 4k+1, for integer $k \geq 1$, or of the form 4k+2, for integer $k \geq 0$.

From these, the following theorem is established:
Theorem 1. For n = 1, the factoriangular number is an even integer. For $n \geq 2$, the factoriangular number is even if n is of the form 4k, for integer $k \geq 1$, or 4k+3, for integer $k \geq 0$; but it is odd if n is of the form 4k+1, for integer $k \geq 1$, or 4k+2, for integer $k \geq 0$.

Proof:

The proof for n = 1 is trivial. If n = 1, then

$$Ft_1 = 1! + T_1 \quad \Leftrightarrow \quad Ft_1 = 1 + \frac{1(1+1)}{2} \quad \Leftrightarrow \quad Ft_1 = 2 \cdot$$

Table 2. Form of n and Parity of Ft_n

n	Form of n	Ft_n	Parity of Ft_n
1	4k+1	2	Even
2	4k+2	5	Odd
3	4k+3	12	Even
4	4k	34	Even
5	4k+1	135	Odd
6	4k+2	741	Odd
7	4k+3	5068	Even
8	4k	40356	Even
9	4k+1	362925	Odd
10	4k+2	3628855	Odd
11	4k+3	39916866	Even
12	4k	479001678	Even
13	4k+1	6227020891	Odd
14	4k+2	87178291305	Odd
15	4k+3	1307674368120	Even
16	4k	20922789888136	Even
17	4k+1	355687428096153	Odd
18	4k+2	6402373705728171	Odd
19	4k+3	121645100408832190	Even
20	4k	2432902008176640210	Even

Hence, Ft_1 is even. For $n \geq 2$, four cases are considered as follows:

Case 1: The natural number n is of the form 4k, for integer k ≥ 1. (Note that if k = 0, then n = 4(0) = 0, which is not included here.)

If n = 4k, then $Ft_{4k} = (4k)! + T_{4k}$. For $n \geq 2$, the factorial of n is even and hence, (4k)! is even and can be written as 2m, for positive integer m. Thus,

$$Ft_{4k} = 2m + \frac{4k(4k+1)}{2}$$

$$\Leftrightarrow \quad Ft_{4k} = 2m + 2k(4k+1)$$

$$\Leftrightarrow \quad Ft_{4k} = 2[m + k(4k+1)] \cdot$$

Therefore, Ft_{4k} is even.

Case 2: The natural number n is of the form 4k+1, for integer k ≥ 1. (Note that if k = 0, then n = 4(0)+1 = 1, which has been considered earlier.)

If n = 4k+1, then $Ft_{4k+1} = (4k+1)! + T_{4k+1}$. Same as in Case 1, (4k+1)! is even and can be written as 2m, for positive integer m. Thus,

$$Ft_{4k+1} = 2m + \frac{(4k+1)(4k+2)}{2}$$

$$\Leftrightarrow \quad Ft_{4k+1} = 2m + \frac{16k^2 + 12k + 2}{2}$$

$$\Leftrightarrow \quad Ft_{4k+1} = 2m + 8k^2 + 6k + 1$$

$$\Leftrightarrow \quad Ft_{4k+1} = 2(m + 4k^2 + 3k) + 1 \cdot$$

Hence, Ft_{4k+1} is odd.

Case 3: The natural number n is of the form 4k+2, for integer k ≥ 0.

If n = 4k+2, then $Ft_{4k+2} = (4k+2)! + T_{4k+2}$. Again, (4k+2)! is even and can take the form 2m, for positive integer m. Thus,

$$Ft_{4k+2} = 2m + \frac{(4k+2)(4k+3)}{2}$$

$$\Leftrightarrow \quad Ft_{4k+2} = 2m + \frac{16k^2 + 20k + 6}{2}$$

$$\Leftrightarrow \quad Ft_{4k+2} = 2m + 8k^2 + 10k + 3$$

$$\Leftrightarrow \quad Ft_{4k+2} = 2(m + 4k^2 + 5k + 1) + 1 \cdot$$

Hence, Ft_{4k+2} is odd.

Case 4: The natural number n is of the form 4k+3, for integer k ≥ 0.

If n = 4k+3, then $Ft_{4k+3} = (4k+3)! + T_{4k+3}$. As in previous cases, (4k+3)! is even and can take the form 2m, for positive integer m. Thus,

$$Ft_{4k+3} = 2m + \frac{(4k+3)(4k+4)}{2}$$

$$\Leftrightarrow \quad Ft_{4k+3} = 2m + \frac{16k^2 + 28k + 12}{2}$$

$$\Leftrightarrow \quad Ft_{4k+3} = 2m + 8k^2 + 14k + 6$$

$$\Leftrightarrow \quad Ft_{4k+3} = 2(m + 4k^2 + 7k + 3) \cdot$$

Therefore, Ft_{4k+3} is even. This completes the proof.

Table 3. Prime Factors of the First 20 Factoriangular Numbers

n	Ft_n	Prime Factors
1	2	2
2	5	5
3	12	$2^2 \cdot 3$
4	34	$2 \cdot 17$
5	135	$3^3 \cdot 5$
6	741	$3 \cdot 13 \cdot 19$
7	5068	$2^2 \cdot 7 \cdot 181$
8	40356	$2^2 \cdot 3^2 \cdot 19 \cdot 59$
9	362925	$3^2 \cdot 5^2 \cdot 1613$
10	3628855	$5 \cdot 557 \cdot 1303$
11	39916866	$2 \cdot 3 \cdot 11 \cdot 604801$
12	479001678	$2 \cdot 3 \cdot 79833613$
13	6227020891	$7^2 \cdot 13 \cdot 9775543$
14	87178291305	$3 \cdot 5 \cdot 7 \cdot 14779 \cdot 56179$
15	1307674368120	$2^3 \cdot 3 \cdot 5 \cdot 10897286401$
16	20922789888136	$2^3 \cdot 29 \cdot 90184439173$
17	355687428096153	$3^2 \cdot 17 \cdot 298373 \cdot 7791437$
18	6402373705728171	$3^2 \cdot 317693 \cdot 2239189583$
19	121645100408832190	$2 \cdot 5 \cdot 19 \cdot 2801 \cdot 228574570001$
20	2432902008176640210	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 59 \cdot 251 \cdot 383 \cdot 2042588183$

Notice that the only prime factoriangular numbers are 2 and 5.

The following theorem is established:

Theorem 2. For $n \geq 3$, Ft_n is composite.

Proof: For $n \geq 3$, Ft_n can be any of the four forms: Ft_{4k}, Ft_{4k+1}, Ft_{4k+2} or Ft_{4k+3}, where $k \geq 1$ for the first three forms and $k \geq 0$ for the last. It was already shown in Theorem 1 that Ft_{4k}, for $k \geq 1$, and Ft_{4k+3}, for $k \geq 0$, are even numbers greater than 2 and therefore composite. Thus, what is needed here is to show that the odd Ft_{4k+1} and Ft_{4k+2} are also factorable into at least two integers, both of which are greater than 1. This was shown as follows: For $k \geq 1$,

$$Ft_{4k+1} = (4k+1)! + \frac{(4k+1)(4k+2)}{2}$$

$$\Leftrightarrow \quad Ft_{4k+1} = (4k+1)(4k)! + (4k+1)(2k+1)$$

$$\Leftrightarrow \quad Ft_{4k+1} = (4k+1)[(4k)! + 2k+1]$$

and

$$Ft_{4k+2} = (4k+2)! + \frac{(4k+2)(4k+3)}{2}$$

$$\Leftrightarrow \quad Ft_{4k+2} = (4k+2)(4k+1)! + (2k+1)(4k+3)$$

$$\Leftrightarrow \quad Ft_{4k+2} = (2k+1)[2(4k+1)! + 4k+3].$$

Hence, Ft_{4k+1} and Ft_{4k+2}, for $k \geq 1$, are both composite and the proof is completed.

The above proof also shows that Ft_{4k+1}, which is odd, is divisible by an odd 4k+1; while Ft_{4k+2}, which is also odd, is divisible by 2k+1 that is equal to an even 4k+2 divided by 2. Hence, the following corollary had already been established also:

Corollary 2.1. An odd Ft_n is divisible by n if n is odd and by n/2 if n is even.

Table 4. Number and Sum of Positive Divisors of the First 20 Factoriangular Numbers

n	Ft_n	$\tau(Ft_n)$	$\sigma(Ft_n)$
1	2	2	3
2	5	2	6
3	12	6	28
4	34	4	54
5	135	8	240
6	741	8	1120
7	5068	12	10192
8	40356	36	109200
9	362925	18	650442
10	3628855	8	4365792
11	39916866	16	87091488
12	479001678	8	958003368
13	6227020891	12	7800884112
14	87178291305	32	159425356800
15	1307674368120	32	3923023104720
16	20922789888136	16	40582997628300
17	355687428096153	24	543994430104008
18	6402373705728171	12	9247902244090848
19	121645100408832190	32	230567740252417440
20	2432902008176640210	256	6831031912334622720

The number of positive divisors, denoted by $\tau(Ft_n)$, and the sum of positive divisors, denoted by $\sigma(Ft_n)$, of each of the first 20 factoriangular numbers were computed also and presented in Table 4.

The sum of proper divisors of each of the first 20 factoriangular numbers was also computed and the results were shown in Table 5, together with the categorization as to whether the factoriangular number is abundant or deficient.

Table 5. The First Few Abundant (A) and Deficient (D) Factoriangular Numbers

N	Ft_n	Sum of Positive Divisors	A / D
1	2	1	D
2	5	1	D
3	12	16	A
4	34	20	D
5	135	105	D
6	741	379	D
7	5068	5124	A
8	40356	68844	A
9	362925	287517	D
10	3628855	736937	D
11	39916866	47174622	A
12	479001678	479001690	A
13	6227020891	1573863221	D
14	87178291305	72247065495	D
15	1307674368120	2615348736600	A
16	20922789888136	19660207740164	D
17	355687428096153	188307002007855	D
18	6402373705728171	2845528538362677	D
19	121645100408832190	108922639843585250	D
20	2432902008176640210	4398129904157982510	A

Notice that there is no perfect number in the list and given the behavior of Ft_n for having not too few divisors and relatively many prime factors as *n* gets larger, it is very likely that the following statement is true:

Conjecture 1. There is no factoriangular number that is a perfect number.

To further characterize factoriangular numbers, the positive divisors of each of the first 15 factoriangular numbers were determined and presented in Table 6.

After examining the positive divisors, another conjecture is hereby stated:

Conjecture 2. Except 2, 5 and 12, the Ft_n for n = 1, 2, 3, respectively, no factoriangular number is a divisor of another factoriangular number.

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Table 6. Positive Divisors of the First 15 Factoriangular Numbers

n	Ft_n	Positive Divisors
1	2	1, 2
2	5	1, 5
3	12	1, 2, 3, 4, 6, 12
4	34	1, 2, 17, 34
5	135	1, 3, 5, 9, 15, 27, 45, 135
6	741	1, 3, 13, 19, 39, 57, 247, 741
7	5068	1, 2, 4, 7, 14, 28, 181, 362, 724, 1267, 2534, 5068
8	40356	1, 2, 3, 4, 6, 9, 12, 18, 19, 36, 38, 57, 59, 76, 114, 118, 171, 177, 228, 236, 342, 354, 531, 684, 708, 1062, 1121, 2124, 2242, 3363, 4484, 6726, 10089, 13452, 20178, 40356
9	362925	1, 3, 5, 9, 15, 25, 45, 75, 225, 1613, 4839, 8065, 14517, 24195, 40325, 72585, 120975, 362925
10	3628855	1, 5, 557, 1303, 2785, 6515, 725771, 3628855
11	39916866	1, 2, 3, 6, 11, 22, 33, 66, 604801, 1209602, 1814403, 3628806, 6652811, 13305622, 19958433, 39916866
12	479001678	1, 2, 3, 6, 79833613, 159667226, 239500839, 479001678
13	6227020891	1, 7, 13, 49, 91, 637, 9775543, 68428801, 127082059, 479001607, 889574413, 6227020891
14	87178291305	1, 3, 5, 7, 15, 21, 35, 105, 14779, 44337, 56179, 73895, 103453, 168537, 221685, 280895, 310359, 393253, 517265, 842685, 1179759, 1551795, 1966265, 5898795, 830269441, 2490808323, 4151347205, 5811886087, 12454041615, 17435658261, 29059430435, 87178291305
15	1307674368120	1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120, 10897286401, 21794572802, 32691859203, 43589145604, 54486432005, 65383718406, 87178291208, 108972864010, 130767436812, 163459296015, 217945728020, 261534873624, 326918592030, 435891456040, 653837184060, 1307674368120

However, looking at the divisors again, it is interesting to see that Ft_n for even n is near or close to a divisor of Ft_{n+1}. For instance, $Ft_8 = 40356$ is close to 40325, a divisor of $Ft_9 = 362925$; $Ft_{10} = 3628855$ to 3628806 of Ft_{11}; $Ft_{12} = 479001678$ to 479001607 of Ft_{13}; and so on. Further experimentations on these reveal the following:

for n = 2,

$$Ft_2 = \frac{Ft_3}{3} + k \Leftrightarrow 5 = \frac{12}{3} + k \Rightarrow k = 1;$$

for n = 4,

$$Ft_4 = \frac{Ft_5}{5} + k \Leftrightarrow 34 = \frac{135}{5} + k \Rightarrow k = 7;$$

for n = 6,

$$Ft_6 = \frac{Ft_7}{7} + k \Leftrightarrow 741 = \frac{5068}{7} + k \Rightarrow k = 17;$$

for n = 8,

$$Ft_8 = \frac{Ft_9}{9} + k \Leftrightarrow 40356 = \frac{362925}{9} + k \Rightarrow k = 31;$$

for n = 10,

$$Ft_{10} = \frac{Ft_{11}}{11} + k \Leftrightarrow 3628855 = \frac{39916866}{11} + k \Rightarrow k = 49;$$

for n = 12,

$$Ft_{12} = \frac{Ft_{13}}{13} + k \Leftrightarrow 479001678 = \frac{6227020891}{13} + k \Rightarrow k = 71;$$

and for n = 14,

$$Ft_{14} = \frac{Ft_{15}}{15} + k \Leftrightarrow 87178291305 = \frac{1307674368120}{15} + k \Rightarrow k = 97$$

.

Similar experimentations on the Ft_n for odd n and the divisors of Ft_{n+1} resulted to the following:

for n = 3,

$$2Ft_3 = \frac{2Ft_4}{4} + k \Leftrightarrow 2(12) = \frac{2(34)}{4} + k \Rightarrow k = 7;$$

for n = 5,

$$2Ft_5 = \frac{2Ft_6}{6} + k \Leftrightarrow 2(135) = \frac{2(741)}{6} + k \Rightarrow k = 23;$$

for n = 7,

$$2Ft_7 = \frac{2Ft_8}{8} + k \Leftrightarrow 2(5068) = \frac{2(40356)}{8} + k \Rightarrow k = 47;$$

for n = 9,

$$2Ft_9 = \frac{2Ft_{10}}{10} + k \Leftrightarrow 2(362925) = \frac{2(3628855)}{10} + k \Rightarrow k = 79;$$

for n = 11,

$$2Ft_{11} = \frac{2Ft_{12}}{12} + k \Leftrightarrow 2(39916866) = \frac{2(479001678)}{12} + k \Rightarrow k = 119$$

and for n = 13,

$$2Ft_{13} = \frac{2Ft_{14}}{14} + k \Leftrightarrow 2(6227020891) = \frac{2(87178291305)}{14} + k \Rightarrow k = 167$$

.

Based on these results, the following theorem is formally stated:

Theorem 3. For even $n \geq 2$, there is a positive integer k such that

$$Ft_n = \frac{Ft_{n+1}}{n+1} + k \cdot$$

For odd $n \geq 3$, there is a positive integer k such that

$$2Ft_n = \frac{2Ft_{n+1}}{n+1} + k \cdot$$

Proof:

For the first part of the theorem,

$$Ft_n = \frac{Ft_{n+1}}{n+1} + k$$

$$\Leftrightarrow \quad k = Ft_n - \frac{Ft_{n+1}}{n+1}$$

$$\Leftrightarrow$$

$$k = \left[n! + \frac{n(n+1)}{2}\right] - \left[\frac{(n+1)! + \frac{(n+1)(n+2)}{2}}{n+1}\right]$$

$$\Leftrightarrow$$

$$k = n! + \frac{n^2+n}{2} - \frac{(n+1)n! + \frac{(n+1)(n+2)}{2}}{n+1}$$

$$\Leftrightarrow \quad k = n! + \frac{n^2+n}{2} - n! - \frac{n+2}{2}$$

$$\Leftrightarrow \quad k = \frac{n^2+n-n-2}{2}$$

$$\Leftrightarrow \quad k = \frac{n^2-2}{2}.$$

If n is even, then n^2 is also even and hence, it can be written that, for positive integer m, $k = (2m-2)/2 = m-1$. Since $n \geq 2$, $n^2 = 2m \geq 4$ or $m \geq 2$, which implies that k is a positive integer.

Similarly, for the second part of the theorem,

$$2Ft_n = \frac{2Ft_{n+1}}{n+1} + k$$

$$\Leftrightarrow \quad k = 2Ft_n - \frac{2Ft_{n+1}}{n+1}$$

$$\Leftrightarrow \quad k = 2\left[n! + \frac{n(n+1)}{2}\right] - \frac{2\left[(n+1)! + \frac{(n+1)(n+2)}{2}\right]}{n+1}$$

$$\Leftrightarrow \quad k = 2n! + (n^2+n) - \frac{2(n+1)n! + (n+1)(n+2)}{n+1}$$

$$\Leftrightarrow \quad k = 2n! + n^2 + n - 2n! - n - 2$$

$$\Leftrightarrow \quad k = n^2 - 2.$$

If n is odd, then n^2 is also odd and hence, it can be written that, for positive integer m, $k = (2m+1) - 2 = 2m-1$. Since $n \geq 3$, $n^2 = 2m + 1 \geq 9$ or $m \geq 4$, which implies that k is a positive integer, more particular, integer $k \geq 7$. This completes the proof.

In the above proof, the following corollaries had also been established:

Corollary 3.1. For even $n \geq 2$,

$$k = Ft_n - \frac{Ft_{n+1}}{n+1} = \frac{n^2-2}{2}$$

and form the sequence {1, 7, 17, 31, 49, 71, 97, 127, 161, …}. For odd $n \geq 3$,

$$k = 2Ft_n - \frac{2Ft_{n+1}}{n+1} = n^2 - 2$$

and form the sequence {7, 23, 47, 79, 119, 167, 223, 287, 359, …}.

Corollary 3.2. For even $n \geq 2$ and $k = (n^2 - 2)/2$, $Ft_n - k$ is a factor of Ft_{n+1}. For odd $n \geq 3$ and $k = n^2 - 2$, $2Ft_n - k$ is a factor of $2Ft_{n+1}$. For both cases, the other factor is $n + 1$.

The Zeckendorf's decompositions of factoriangular numbers were also included here. These are given in Table 7, where F_k stands for Fibonacci number.

Table 7. Zeckendorf's Decomposition of the First 15 Factoriangular Numbers

n	Ft_n	Zeckendorf's Decomposition
1	2	F_3
2	5	F_5
3	12	$F_6 + F_4 + F_2$
4	34	F_9
5	135	$F_{11} + F_9 + F_6 + F_4 + F_2$
6	741	$F_{15} + F_{11} + F_9 + F_6$
7	5068	$F_{19} + F_{15} + F_{13} + F_9 + F_6 + F_3$
8	40356	$F_{23} + F_{21} + F_{15} + F_{11} + F_9 + F_7 + F_5 + F_3$
9	362925	$F_{28} + F_{23} + F_{21} + F_{19} + F_{16} + F_{13} + F_{11} + F_8$
10	3628855	$F_{33} + F_{25} + F_{23} + F_{14} + F_{12} + F_{10} + F_7 + F_5 + F_2$
11	39916866	$F_{38} + F_{29} + F_{27} + F_{25} + F_{23} + F_{21} + F_{18} + F_{15} + F_{12} + F_{10} + F_8 + F_6$
12	479001678	$F_{43} + F_{38} + F_{34} + F_{29} + F_{27} + F_{19} + F_{16} + F_{13} + F_{11} + F_9 + F_7 + F_2$
13	6227020891	$F_{48} + F_{45} + F_{42} + F_{36} + F_{31} + F_{28} + F_{25} + F_{20} + F_{12} + F_{10} + F_8 + F_5 + F_3$
14	87178291305	$F_{43} + F_{38} + F_{34} + F_{29} + F_{27} + F_{19} + F_{16} + F_{13} + F_{11} + F_9 + F_7 + F_2$
15	1307674368120	$F_{59} + F_{56} + F_{54} + F_{52} + F_{48} + F_{44} + F_{42} + F_{40} + F_{32} + F_{29} + F_{26} + F_{24} + F_{21} + F_{16} + F_{14} + F_{12} + F_{10} + F_8 + F_3$

Given the increasing number of terms in the Zeckendorf's decomposition as Ft_n gets larger, the following statements are believed to be true:

Conjecture 3. $Ft_1 = 2$, $Ft_2 = 5$ and $Ft_4 = 34$ are the only factoriangular numbers that are also Fibonacci numbers.

Conjecture 4. There is no factoriangular number that has a Zeckendorf's decomposition of only two terms.

Conjecture 5. Only $Ft_3 = 12$, $Ft_6 = 741$, $Ft_5 = 135$ and $Ft_7 = 5068$ has a Zeckendorf's decomposition of only 3, 4, 5 and 6 terms, respectively.

To add some other minor characteristics of factoriangular numbers, the end digits and digital roots were also examined. The characterizations are as follows:

For $5 \leq n \leq 24$, Ft_n ends in digit 5, 1, 8, 6, 5, 5, 6, 8, 1, 5, 0, 6, 3, 1, 0, 0, 1, 3, 6, 0, respectively and this cycle or periodic sequence of 20 Ft_n unit digits repeats for $25 \leq n \leq 44$, $45 \leq n \leq 64$, and so on. Another noticeable on the said sequence of end digits is that the first set of five digits are the reverse of the second set of five digits while the third set of five digits are the reverse of the fourth set of five digits. For $n = 1, 2, 3, 4$, the Ft_n unit digits are 2, 5, 2 and 4, respectively.

For $6 \leq n \leq 14$, the digital roots of Ft_n are 3, 1, 9, 9, 1, 3, 6, 1, 6, respectively and this cycle or periodic sequence of 9 Ft_n digital roots repeats for $15 \leq n \leq 23$, $24 \leq n \leq 32$, and so on. For $n = 1, 2, 3, 4, 5$, the Ft_n digital roots are 2, 5, 3, 7 and 9, respectively.

Table 8. End Digit and Digital Root of the First 30 Factoriangular Numbers

n	End Digit of Ft_n (in bold font)	Digital Root
1	**2**	2
2	**5**	5
3	1**2**	3
4	3**4**	7
5	13**5**	9
6	74**1**	3
7	506**8**	1
8	4035**6**	9
9	36292**5**	9
10	362885**5**	1
11	3991686**6**	3
12	47900167**8**	6
13	622702089**1**	1
14	8717829130**5**	6
15	130767436812**0**	3
16	2092278988813**6**	1
17	35568742809615**3**	9
18	640237370572817**1**	9
19	12164510040883219**0**	1
20	243290200817664021**0**	3
21	5109094217170944023**1**	6
22	112400072777760768025**3**	1
23	2585201673888497664027**6**	6
24	62044840173323943936030**0**	3
25	1551121004333098598400032**5**	1
26	40329146112660563558400035**1**	9
27	1088886945041835216076800037**8**	9
28	30488834461171386050150400040**6**	1
29	884176199373970195454361600043**5**	3
30	265252859812191058636308480000046**5**	6

CONCLUSIONS

The term *factoriangular number* was introduced here to name a number resulting from adding corresponding factorial and triangular number. This sequence of natural numbers has interesting properties or characteristics, some of which had been established in this study. In particular, the parity of factoriangular number is as follows: For $n = 1$, the factoriangular number is even; for $n \geq 2$, the factoriangular number is even if n is of the form 4k, for integer $k \geq 1$, or 4k+3, for integer $k \geq 0$ but it is odd if n is of the form 4k+1, for integer $k \geq 1$, or 4k+2, for integer $k \geq 0$. Also, for $n \geq 3$, the factoriangular number is composite, and it is divisible by n if n is odd and by $n/2$ if n is even.

It had been shown also that for even $n \geq 2$, there is a positive integer k such that $Ft_n = [Ft_{n+1} / (n+1)] + k$ and this k is equal to $(n^2 - 2)/2$, while for odd $n \geq 3$, there is a positive integer k such that $2Ft_n = [2Ft_{n+1} / (n+1)] + k$ and this k is equal to $n^2 - 2$.

More experimentation may be done to further characterize factoriangular numbers. Proving the conjectures presented here is also suggested.

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