Investigating Triangular Numbers with greatest integer function, Sequences and Double Factorial

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Abstract - The nth Triangular number denoted by T_n is defined as the sum of the first n consecutive positive integers. A positive integer n is a Triangular Number if and only if $\boldsymbol{T_n} = \frac{\boldsymbol{n(n+1)}}{\boldsymbol{2}}$ [1]. We stated and proved a sequence of positive integers (A, B, C) is consecutive triangular numbers if and only if $\sqrt{\boldsymbol{B+C}} - \sqrt{\boldsymbol{B+A}} = \mathbf{1}$ and $B - \boldsymbol{A} = \sqrt{\boldsymbol{B+A}}$. We consider a ceiling function $\left\lceil \frac{x}{2} \right\rceil$ to state and prove a necessary and sufficient condition for a number $\boldsymbol{m} = \boldsymbol{T_n} = \left\lceil \frac{\boldsymbol{n+1}}{\boldsymbol{2}} \right\rceil (\mathbf{2}\left\lceil \frac{\boldsymbol{n}}{\boldsymbol{2}} \right\rceil + \mathbf{1})$ to be a triangular number for each $n \geq 0$. A formula to find $\boldsymbol{lcm}$ and $\boldsymbol{gcd}$ of any two consecutive triangular numbers and a double factorial is introduced to find products of triangular numbers.

Key words: *Triangular numbers, ceiling function, double factorial.*

INTRODUCTION

A triangular number T_n is a number of the form $T_n = 1 + 2 + 3 + \cdots + n$, where n is a natural number. So that the first few triangular numbers are $1, 3, 6, 10, 15, 21, 28, 36, 45, \ldots$ [2]. A well-known fact about triangular numbers is that y is a triangular number if and only if $8y + 1$ is a perfect square [1]. Triangular numbers can be thought of as the numbers of dots that can be arranged in the shape of a square.

Lemma 0.0.1: A positive integer m is triangular if and only if it is in the form of $m = \sum_{i=1}^{n} \frac{i(i+1)}{2}$ for $n \geq 1$.

Theorem 0.0.2: For any integer n, $\left\lceil \frac{n}{2} \right\rceil = \begin{cases} \frac{n}{2} \;; & if\ n\ is\ even \\ \frac{n+1}{2} \;; & if\ n\ is\ odd \end{cases}$

Theorem 0.0.3: A positive integer m is triangular if and only if

$$m = \boldsymbol{T_n} = \left\lceil \frac{\boldsymbol{n+1}}{\boldsymbol{2}} \right\rceil \left(\mathbf{2}\left\lceil \frac{\boldsymbol{n}}{\boldsymbol{2}} \right\rceil + \mathbf{1}\right) \text{ for each } n \geq 0.$$

Proof: $(\Longrightarrow)$ Suppose a positive integer m is triangular. There exist $n \geq 1$ such that $m = \frac{n(n+1)}{2}$, (Lemma 0.0.1).

Case 1: When n is odd. If n Is odd then $\frac{n+1}{2} = \left\lceil \frac{\boldsymbol{n+1}}{\boldsymbol{2}} \right\rceil$ and $\left\lceil \frac{\boldsymbol{n}}{\boldsymbol{2}} \right\rceil = \frac{\boldsymbol{n+1}}{\boldsymbol{2}}$. The later implies $\boldsymbol{n} + \mathbf{1} = \mathbf{2}\left\lceil \frac{\boldsymbol{n}}{\boldsymbol{2}} \right\rceil$ and $\boldsymbol{n} + \mathbf{2} = (\mathbf{2}\left\lceil \frac{\boldsymbol{n}}{\boldsymbol{2}} \right\rceil + \mathbf{1})$. Therefore m= $\left(\frac{\boldsymbol{n+1}}{\boldsymbol{2}}\right)(\boldsymbol{n}+\mathbf{2}) = \left\lceil \frac{\boldsymbol{n+1}}{\boldsymbol{2}} \right\rceil \left(\mathbf{2}\left\lceil \frac{\boldsymbol{n}}{\boldsymbol{2}} \right\rceil + \mathbf{1}\right)$.

Case 2: When n is even. If n is even then $\left\lceil \frac{\boldsymbol{n}}{\boldsymbol{2}} \right\rceil = \frac{\boldsymbol{n}}{\boldsymbol{2}}$. This implies $n = 2\left\lceil \frac{n}{2} \right\rceil$ and $n + 1 = 2\left\lceil \frac{n}{2} \right\rceil + 1$.
Similarly for n is even $\frac{n+2}{2} = \left\lceil \frac{n+1}{2} \right\rceil$. Combining the former and the later we have

$$m = (n+1)\left(\frac{n+2}{2}\right) = \left\lceil \frac{n+1}{2} \right\rceil \left(2\left\lceil \frac{n}{2} \right\rceil + 1\right).$$

$(\Longleftarrow)$ Suppose $\boldsymbol{m} = \boldsymbol{T_n} = \left\lceil \frac{\boldsymbol{n+1}}{\boldsymbol{2}} \right\rceil \left(\mathbf{2}\left\lceil \frac{\boldsymbol{n}}{\boldsymbol{2}} \right\rceil + \mathbf{1}\right)$ & is even for some $n \geq 0$. We show that m is triangular. Set $A = \left\lceil \frac{\boldsymbol{n+1}}{\boldsymbol{2}} \right\rceil$ and $B = \mathbf{2}\left\lceil \frac{\boldsymbol{n}}{\boldsymbol{2}} \right\rceil + \mathbf{1}$. Then either A and B are both even or they have different parity. But because B is always odd , A must be even.

Consider $B = \mathbf{2}\left\lceil\frac{n}{2}\right\rceil + \mathbf{1}$ is odd**.** Then $\left\lceil\frac{n}{2}\right\rceil$ is either even or odd. Suppose it is odd. This implies n is odd. Therefore $\left\lceil\frac{n}{2}\right\rceil = \frac{n+1}{2}$

and $\left\lceil\frac{n+1}{2}\right\rceil = \frac{n+1}{2}$ **.** From the former $2\left\lceil\frac{n}{2}\right\rceil + 1 = 2\left(\frac{n+1}{2}\right) + 1 = n + 2$ and combining with the later**,**

$$\boldsymbol{m} = \boldsymbol{T_n} = \left\lceil\frac{n+1}{2}\right\rceil\left(\mathbf{2}\left\lceil\frac{n}{2}\right\rceil + \mathbf{1}\right) = \frac{(n+1)(n+2)}{2}$$ **.** Hence by (Lemma 0.0.1) m is triangular.

Suppose $\left\lceil\frac{n}{2}\right\rceil$ is even. Then either n is even or odd. Suppose n is even. Then we have $\left\lceil\frac{n+1}{2}\right\rceil = \frac{n+2}{2}$ and $\left\lceil\frac{n}{2}\right\rceil = \frac{n}{2}$. Hence $\left(\mathbf{2}\left\lceil\frac{n}{2}\right\rceil + \mathbf{1}\right) = \mathbf{2}\left(\frac{n}{2}\right) + 1 = n + 1$ and therefore,

$$\boldsymbol{m} = \boldsymbol{T_n} = \left\lceil\frac{n+1}{2}\right\rceil\left(\mathbf{2}\left\lceil\frac{n}{2}\right\rceil + \mathbf{1}\right) = \frac{(n+1)(n+2)}{2}$$ is triangular.

Similarly when n is odd, we have $\left\lceil\frac{n+1}{2}\right\rceil = \frac{n+1}{2}$ **and** $\left(2\left\lceil\frac{n}{2}\right\rceil + 1\right) = n + 2$ and hence

$$\boldsymbol{m} = \boldsymbol{T_n} = \left\lceil\frac{n+1}{2}\right\rceil\left(\mathbf{2}\left\lceil\frac{n}{2}\right\rceil + \mathbf{1}\right) = \frac{(n+1)(n+2)}{2}$$ is triangular.

In similar fashion one can prove the case $m = T_n = \left\lceil\frac{n+1}{2}\right\rceil\left(2\left\lceil\frac{n}{2}\right\rceil + 1\right)$ & is odd for some $n \geq 0$. ■

Theorem 0.0.4:

A sequence of positive integers in the order (A, B, C) is consecutive triangular numbers if and only if

$$\sqrt{\boldsymbol{B}+\boldsymbol{C}} - \sqrt{\boldsymbol{B}+\boldsymbol{A}} = \mathbf{1} \qquad (*)$$

and

$$B - \boldsymbol{A} = \sqrt{\boldsymbol{B}+\boldsymbol{A}}\,. \qquad (**)$$

Proof. (⇒) Let (A, B, C) be a sequence of positive integers in the order. Suppose $\sqrt{B + C} + \sqrt{B + A} = 1$ and $B - \boldsymbol{A} = \sqrt{\boldsymbol{B}+\boldsymbol{A}}$**.**

From the later when we square both sides, $(B - A)^2 = B + A\ldots$ (***)

and combining the former with (***) we have $\sqrt{B + C} = 1 + \sqrt{B + A} = 1 + \sqrt{(B - A)^2}$

This implies $\sqrt{B + C} = 1 + |B - A| = 1 + B - A$ because $B > A$ (****).

Squaring both sides of (****) gives, $B + \boldsymbol{C} = (\mathbf{1} + \boldsymbol{B} - \boldsymbol{A})^{\mathbf{2}}$**.** Let $B - A = \boldsymbol{n}$, for some $\boldsymbol{n} \in \mathbb{Z}^+$**.** This implies $B + \boldsymbol{C} = (\mathbf{1} + \boldsymbol{n})^{\mathbf{2}}$ and from (***) $B + A = n^2$.

Hence $\sqrt{\boldsymbol{B}+\boldsymbol{C}} - \sqrt{\boldsymbol{B}+\boldsymbol{A}} = \mathbf{1}$ is true if and only if $B + C = (n + 1)^2$ and $B + A = n^2$ for some $n \geq 0$.

Therefore, $B = n^2 - A$ and $C - A = 2n + 1$. This implies $C = 2n + 1 + A$.

Consider the sequence

$$(A, B, C) = (A, n^2 - A, 2n + 1 + A), \qquad (*****).$$

From $(**)$, $\boldsymbol{B} - \boldsymbol{A} = \boldsymbol{n}$**.** Combining $(**)$ and $(***)$, we have $\boldsymbol{n}^{\mathbf{2}} - \boldsymbol{n} = \mathbf{2}\boldsymbol{A}$**,** which implies

$\boldsymbol{A = \frac{n^2-n}{2} = \frac{(n-1)n}{2}}$ **and**

$\boldsymbol{C = 2n+1+A = 2n+1+\frac{n^2-n}{2} = \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2}}$ and

B $\boldsymbol{= n^2 - A = n^2 - \frac{n^2-n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}}$.

Therefore (A , B , C) $= (\frac{n^2-n}{2}, \frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}) = (T_{n-1}, T_n, T_{n+1})$ is a sequence of consecutive triangular numbers.

(⇐) Suppose a sequence of integers (A, B, C) is consecutive triangular numbers.

Set $A = T_m$. Then $B = T_{m+1}$ and $C = T_{m+2}$. By (Lemma 0.0.1),

$A = \frac{m(m+1)}{2}$, $B = \frac{(m+1)(m+2)}{2}$ and $C = \frac{(m+2)(m+3)}{2}$.

This implies $B + C = (m+2)^2$ and $B + A = (m+1)^2$. Thus

$$\sqrt{B+C} - \sqrt{B+A} = \sqrt{(m+2)^2} - \sqrt{(m+1)^2}$$

$$= |m+2| - |m+1| = 1 \quad \text{and ,} \qquad (\Delta)$$

B – A = $\frac{(m+1)(m+2)}{2} - \frac{m(m+1)}{2} = m+1$ and

$$\sqrt{B+A} = \sqrt{\frac{(m+1)(m+2)}{2} + \frac{m(m+1)}{2}} = \sqrt{(m+1)^2} = |m+1| = m+1 .$$

Therefore $B - A = \sqrt{B+A}$. $(\Delta\Delta)$

From (Δ) and $(\Delta\Delta)$ if a sequence of integers (A, B, C) is consecutive triangular numbers,

then $\sqrt{B+C} - \sqrt{B+A} = 1$ and $B - A = \sqrt{B+A}$. ■

Note: For any $k \geq 1$ the number $n = 2^{k-1}(2^k - 1)$ is triangular in particular if $(2^k - 1)$ is prime for $k > 1$ then $n = 2^{k-1}(2^k - 1)$ is perfect and also triangular number. To investigate the converse i.e., (in our next paper) which even triangular numbers has the form of $n = 2^{k-1}(2^k - 1)$ and are perfect we explore the followings.

Definition 0.0.5: The greatest common integer d that divides two non-zero integers a and b is called the **greatest common divisor** of a and b, denoted by $\gcd(a, b)$.

Example 0.0.6: Given $x = p_1^m p_2^a$ and $y = p_1^n p_2^b$ where p_1 and p_2 are distinct primes, the

$$\mathbf{gcd}(\mathbf{x}, \mathbf{y}) = \mathbf{p_1^{\min(n,m)} p_2^{\min(a,b)}}$$

Definition 0.0.7: The least common multiple of the integers a and b is called the **smallest positive integer** that is divisible by both a and b, denoted by $\operatorname{lcm}(a, b)$.

Example 0.0.8: Given $x = p_1^m p_2^a$ and $y = p_1^n p_2^b$ where p_1 and p_2 are distinct primes the

$$\mathbf{lcm}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{p_1^{\max(n,m)} p_2^{\max(a,b)}}$$

Theorem 0.0.9 [4,5]: For two positive integers a and b, $\boldsymbol{ab = lcm\ (a\ , b)\ \mathrm{gcd}(a\ , b)}$.

Example 0.0.10: Given $x = p_1^m p_2^a$ and $y = p_1^n p_2^b$ where p_1 and p_2 are primes, then
$\mathbf{xy} = p_1^m p_2^a\ p_1^n p_2^b\ =\ \mathbf{gcd}(\boldsymbol{x}, \boldsymbol{y})\ \boldsymbol{lcm}(\boldsymbol{x}\ , \boldsymbol{y}) =\ \boldsymbol{p_1^{\min(n\ ,m)} p_2^{\min(a,b)} p_1^{\max(n\ ,m)} p_2^{\max(a,b)}}$

Theorem 0.0.11:

For each $n \geq 1$, $\big(f(n), g(n)\big) = (\,T_{4n-1}\ , T_{4n}\,)$ and $\big(\phi(n), \eta(n)\big) = (\,T_{4n-3}\ , T_{4n-2}\,)$ are the set of ordered pairs with

consecutive even and consecutive odd triangular numbers.

Note: See the table at page 9 below.

Theorem 0.0.12:

$$\begin{cases} \mathrm{gcd}\big(f(n), g(\,n)\big) = \ 2n \\ \mathrm{gcd}\big(\phi(n), \eta(n)\big) = 2n - 1 \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{lcm}\big(f(n), g(\,n)\big) \ = \ 3\binom{4n+1}{3} \\ \mathrm{lcm}\big(\phi(n), \eta(n)\big) \ = \ 3\binom{4n-1}{3} \end{cases}$$

Proof:

$f(n) =\ T_{4n-1} = \frac{(4n-1)(4n)}{2} = (2n)(4n-1)$ and $g(n) = T_{4n} = \frac{(4n)(4n+1)}{2} = \ (2n)(4n+1)$.

If $d|\ (4n-1)$ and $d|(4n+1)$ then $|(4n+1) - (4n-1)$. This implies $d|2$ and then $d|1$

or $d|2$. But $d \neq 2$, because d is a divisor of an odd integer. Therefore the only divisor of

$(4n+1)$ and $(4n-1)$ is 1. Hence the $\mathrm{gcd}(\,4n-1, 4n+1) = 1$. (⋄⋄⋄)

Therefore for each n , $f(n) =\ T_{4n-1}$ and $g(n) = T_{4n}$ $\mathrm{gcd}(\,f(n), g(n)) = \mathbf{2}\boldsymbol{n}$ and then

$$lcm\ \big(\,f(n), g(n)\big) = \frac{\boldsymbol{f(n)g(n)}}{\mathbf{gcd}(\,\boldsymbol{f(n)}, \boldsymbol{g(n)})} \ = \ \frac{(2n)(4n-1)(2n)(4n+1)}{2n}$$

$$= (2\mathrm{n})((4\mathrm{n}-1)(4\mathrm{n}+1)\ = \ \frac{1}{2n}(\,T_{4n-1}T_{4n})$$

$$= \frac{1}{2n}\binom{4n}{2}\binom{4n+1}{2} = \ 3\binom{4n+1}{3}\ .$$

Next we find $\mathrm{lcm}\big(\,\phi(n), \eta(\,n)\big)$ and $\mathrm{gcd}\big(\,\phi(n), \eta(\,n)\big)$.

$$\phi(n) =\ T_{4n-3} \ = \ \frac{(4n-3)(4n-2)}{2} = (4n-3)(2n-1)$$

and

$\eta(n) =\ T_{4n-2} \ = \ \frac{(4n-2)(4n-1)}{2} \ = \ (4n-1)(2n-1)$. The $\mathrm{gcd}(\,4n-1, 4n-3) = 1$. (⋄⋄⋄) above.

Therefore, $\mathrm{gcd}\big(\,\phi(n), \eta(\,n)\big) = \mathrm{gcd}((\,4n-3)\ (2n-1),\ (4n-1)(2n-1)) = 2n-1$.

By (Theorem 0.0.8), $lcm\ \big(\,\phi(n), \eta(n)\big) = \frac{\phi(n)\eta(n)}{\mathrm{gcd}(\,\phi(n), \eta(n))} \ = \ \frac{(2n-1)(4n-3)(4n-1)(2n-1)}{2n-1}$

$$= (2n-1)(4n-1)(4n-3) = \frac{1}{(n-1)}(T_{4n-3}T_{2n-2})$$

$$= \frac{1}{2n}\binom{4n-2}{2}\binom{2n-1}{2} = 3\binom{4n-1}{3}$$ ■

Example 0.0.13: Find $\gcd(T_7, T_8)$ and $\operatorname{lcm}(T_7, T_8)$.

Answer: $T_7 = T_{4n-1} = 28$ and $T_8 = T_{4n} = 36$ where $n = 2$. Therefore

$\gcd(T_7, T_8) = \gcd(28, 36) = 2n = 4$ and $\operatorname{lcm}(T_7, T_8) = 3\binom{9}{3} = 252 = \frac{(28)(36)}{2}$.

Theorem 0.0.14:

Define a sequence

$F_n = \sum_{i=0}^{n}(4i+1)$ and $G_n = \sum_{i=0}^{n}(4i+3)$. Then

$$\sum_{i=1}^{2n} T_i = \sum_{i=0}^{n-1}\sum_{k=0}^{i}(F_i + G_i) \quad .$$

Proof: *Given*

$F_t = \sum_{k=0}^{t}(4k+1)$ *and* $G_t = \sum_{k=0}^{t}(4k+3)$. *Then*

$$\sum_{i=1}^{2n} T_{2i} = \sum_{i=0}^{n-1}\sum_{k=0}^{i}(F_i + G_i) \qquad (\odot\odot)$$

We use induction to prove the statement. We verify it is true for $n = 1$. The left side of

$(\odot\odot)$ $\sum_{i=1}^{2} T_i = T_1 + T_2 =$ 1+3 = 4 and the right side $\sum_{i=0}^{0}\sum_{k=0}^{0}(F_i + G_i) = F_0 + G_0 = 1 + 3 = 4$.

Let $t \in \mathbb{Z}^+$ and suppose the statement in $(\odot\odot)$ is true for $n = t$ that is

$\sum_{i=1}^{2t} T_{2i} = \sum_{i=0}^{t-1}\sum_{k=0}^{i}(F_i + G_i)$. Now we show that it is true for $n = t + 1$. Thus

$\sum_{i=1}^{2(t+1)} T_{2i} = \sum_{i=1}^{2t+2} T_{2i} = \sum_{i=1}^{2t} T_{2i} + T_{2t+1} + T_{2t+2}$, but

$\mathrm{F_t} = \sum_{\mathrm{k=1}}^{\mathrm{t}}(4\mathrm{k}+1) + 1 = \frac{4\mathrm{t}(\mathrm{t}+1)}{2} + \mathrm{t} + 1 = (\mathrm{t}+1)(2\mathrm{t}+1) = \mathrm{T_{2t+1}}$, and

$\mathrm{G_t} = \sum_{\mathrm{k=1}}^{\mathrm{t}}(4\mathrm{k}+3) + 3 = \frac{\mathrm{t}(\mathrm{t}+1)}{2} + +3\mathrm{t} + 3 = (\mathrm{t}+1)(2\mathrm{t}+3) = \mathrm{T_{2t+2}}$. Hence,

$\mathrm{T_{2t+1}} = \mathrm{F_t}$ and $\mathrm{T_{2t+2}} = \mathrm{G_t}$ and $\sum_{i=1}^{2(t+1)} T_{2i} = \sum_{\mathrm{i=1}}^{\mathrm{2t}} \mathrm{T_{2i}} + \mathrm{F_t} + \mathrm{G_t}$ and therefore

$$\sum_{i=1}^{2(t+1)} T_{2i} = \sum_{i=1}^{2t+2} T_{2i} = \sum_{i=1}^{2t} T_{2i} + T_{2t+1} + T_{2t+2}$$

$$= \sum_{i=0}^{t-1}\sum_{k=0}^{i}(F_i + G_i) + F_t + G_t$$

$$= \sum_{i=0}^{t-1}\sum_{k=0}^{i}(F_i + G_i) + \sum_{k=0}^{t}(4k+1) + \sum_{k=0}^{t}(4k+3)$$

$= \sum_{i=0}^{t}\sum_{k=0}^{i}(F_i + G_i)$ and the statement is true for $n = t + 1$.

Hence $\sum_{i=1}^{2n} T_i = \sum_{i=0}^{n-1}\sum_{k=0}^{i}(F_i + G_i)$ ■

Theorem 0.0.15: For each $n \geq 1$,

$$\sum_{i=1}^{n} \boldsymbol{T_i^2} = \frac{\boldsymbol{n}}{\boldsymbol{60}} \boldsymbol{T_{2n+1}} \binom{\boldsymbol{3T_n+2}}{\boldsymbol{3T_n+1}} + \frac{\boldsymbol{1}}{\boldsymbol{2}} \boldsymbol{T_n^2}$$

Example 0.0.16: Find $\sum_{i=1}^{3} T_i^2$.

Answer: $\sum_{i=1}^{3} T_i^2 = T_1^2 + T_2^2 + T_3^2 = 1^2 + 3^2 + 6^2 = 1 + 9 + 36 = 46$ and $\frac{3}{60} T_7 \binom{3T_3+2}{3T_3+1} + \frac{1}{2} T_3^2 = \frac{3}{60}.28.\binom{20}{19} + \frac{1}{2}(36) = \frac{3}{60}.28.20 + \frac{1}{2}(36) = 28 + 18 = 46$.

This implies $\sum_{i=1}^{3} T_i^2 = 46 = \frac{3}{60} T_7 \binom{3T_3+2}{3T_3+1} + \frac{1}{2} T_3^2$.

Proof: We use the following identities: (⊗)

1) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
2) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$
3) $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)}{30}(3n^2 + 3n - 1)$

For each $n \geq 1$, ${T_n}^2 - {T_{n-1}}^2 = n^3$. This implies

$$\sum_{i=1}^{n}\left({T_i}^2 - {T_{i-1}}^2\right) = \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2 = T_n^2 .$$ Hence

$$T_k^2 = \sum_{i=1}^{k} i^3 \quad \text{and} \quad \sum_{k=1}^{n} T_k^2 = \sum_{k=1}^{n}\sum_{i=1}^{k} i^3 = \sum_{k=1}^{n}\frac{k^2(k+1)^2}{4} = \frac{1}{4}\sum_{k=1}^{n}(k^4 + 2k^3 + k^2) . \qquad (⊕⊕)$$

But,

$$\begin{aligned}
\sum_{k=1}^{n} k^4 + \sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} k^4 - \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k^2 \\
&= \frac{n(n+1)(2n+1)}{30}(3n^2 + 3n - 1) - \frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)(2n+1)}{6} \\
&= \frac{n(n+1)(2n+1)}{6}\left(\frac{3n^2+3n-1}{5} - 1\right) + 2\frac{n(n+1)(2n+1)}{6} \\
&= \frac{(n-1)n(n+1)(n+2)(2n+1)}{10} + \frac{n(n+1)(2n+1)}{3} \\
&= n(n+1)(2n+1)\left(\frac{(n-1)(n+2)}{10} + \frac{1}{3}\right) \\
&= \frac{1}{30}\mathrm{n(n+1)(2n+1)(3n(n+1)+4)} \\
&= \frac{n}{30}\frac{(2n+1)(2n+2)}{2}(3n(n+1)+4) \\
&= \frac{n}{30} T_{2n+1}\left(\frac{6n(n+1)}{2} + 4\right) = \frac{n}{30} T_{2n+1}(6T_n + 4) \\
&= \frac{n}{15} T_{2n+1}(3T_n + 2) \qquad (⊕⊕⊕)
\end{aligned}$$

Combining (⊕) and (⊕⊕) we have,

$$\begin{aligned}
\frac{1}{4}\sum_{k=1}^{n}(k^4 + 2k^3 + k^2) &= \frac{1}{4}\left(\sum_{k=1}^{n} k^4 + \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k^3\right) \\
&= \frac{1}{4}\left(\frac{n}{15} T_{2n+1}(3T_n + 2) + 2\sum_{k=1}^{n} k^3\right) \qquad \text{(see (⊕⊕⊕))} \\
&= \frac{1}{4}\left(\frac{n}{15} T_{2n+1}(3T_n + 2) + 2\frac{n^2(n+1)^2}{4}\right) \qquad \text{(see (⊗))} \\
&= \frac{n}{60} T_{2n+1}(3T_n + 2) + \frac{1}{2} T_n^2 \\
&= \frac{n}{60} T_{2n+1}\binom{3T_n+2}{3T_n+1} + \frac{1}{2} T_n^2
\end{aligned}$$

Hence for each for each $n \geq 1$,

$$\sum_{i=1}^{n} T_i^2 = \frac{n}{60} T_{2n+1}\binom{3T_n+2}{3T_n+1} + \frac{1}{2} T_n^2 \qquad ■$$

Double Factorial

The product of the integers from 1 up to some non-negative integers n that have the same parity as n is called double factorial or semi factorial of n and is denoted by $n!!$ [3, 6]. That is

$$\boldsymbol{n!! = \prod_{k=0}^{m}(n - 2k) = n(n-2)(n-4)\ldots}, \text{ where } m = \left\lceil\frac{n}{2}\right\rceil - 1 .$$

A consequence of this definition is that $0!! = 1$. For even n, the double factorial is

$$n!! = \prod_{k=1}^{\frac{n}{2}}(2k) = n(n-2)\ldots 2 \quad \text{and} \quad \text{for odd } n,$$

$$n!! = \prod_{k=1}^{\frac{n+1}{2}}(2k-1) = n(n-2)\ldots .1 .$$

Theorem 0.0.17:
Let T_n be the nth triangular number. Then for $p \geq 1$,

$$(2p+1)!! = \frac{1}{p!}\prod_{i=1}^{p} T_{2i}.$$

Example 0.0.18:
5!! = (2.2+1)!! = 1 . 3 . 5 = 15 = $\frac{1}{2!}\prod_{i=1}^{2} T_{2i} = \frac{1}{2} .T_2.T_4 = \frac{1}{2!}$(3 .10) = 15 and

7!! = (2.3+1)!! = 1 . 3 . 5. 7= 105 = $\frac{1}{3!}\prod_{i=1}^{3} T_{2i} = \frac{1}{3!} .T_2.T_4.T_6 = \frac{1}{6}$(3 .10 .21) = 105 .

Proof: We prove by induction. Let $P(p)$ be the statement that

$$(2p+1)!! = \frac{1}{p!}\prod_{i=1}^{p} T_{2i}. \qquad (\circ\circ\circ)$$

We verify that $P(1)$ is true. When $p = 1$, the left side of $(\circ\circ\circ)$ $(2.1+1) = 3!! = 3$ and the right side $\frac{1}{1!}\prod_{i=1}^{1} T_{2i} = T_2$ = 3 = 3!! = 1. 3, so both sides are equal and $P(1)$ is true.

Let $k \in \mathbb{Z}^+$ and suppose $P(k)$ is true for $n = k$, i.e., $(2k+1)!! = \frac{1}{k!}\prod_{i=1}^{k} T_{2i}.$ $\qquad (\circ\circ\circ\circ)$
Next we show that
$P(k+1)$ is true for each $k \geq 1$ that is $(2(k+1)+1)!! = \frac{1}{(k+1)!}\prod_{i=1}^{k+1} T_{2i}.$

$(2(k+1)+1)!! = (2k+3)!! = (2k+3)(2k+1)!!$

$= (2k+3)\frac{1}{k!}\prod_{i=1}^{k} T_{2i}$ **(See $(\circ\circ\circ\circ)$)**

$= \frac{1}{k!}\prod_{i=1}^{k} T_{2i}$ (2k+3) $= \frac{k+1}{(k+1)!}\prod_{i=1}^{k} T_{2i}$ (2k+3) (Because $\frac{1}{k!} = \frac{k+1}{(k+1)!}$)

$= \frac{k+1}{(k+1)!}\prod_{i=1}^{k} T_{2i}$ (2k+3) $= \frac{1}{(k+1)!}\prod_{i=1}^{k} T_{2i}\ (2k+3)\ (k+1)$

But $T_{2k+2} = \frac{(2k+2)(2k+3)}{2}$, **Lemma (0.0.1)** which implies $T_{2k+2} = \frac{(2k+2)(2k+3)}{2} = (2k+3)(k+1)$.

Consequently, $2(k+1)+1)!! = (2k+3)!! = \frac{1}{(k+1)!}\prod_{i=1}^{k} T_{2i}\ (2k+3)\ (k+1)$

$= \frac{1}{(k+1)!}\prod_{i=1}^{k} T_{2i} .\quad T_{2k+2}$

$= \frac{1}{(k+1)!}\prod_{i=1}^{k+1} T_{2i} . = P(k+1)$

This implies $P(k+1)$ is true for each $k \geq 1$, and hence ,

$$(2p+1)!! = \frac{1}{p!}\prod_{i=1}^{p} T_{2i} \text{ for each } p \geq 1. \qquad \blacksquare$$

ODD and EVEN Triangular Numbers with Corresponding Subscripts.

1	3	6	10	15	21	28	36	45	55
66	78	91	105	120	136	153	171	190	210
231	253	276	300	325	351	378	406		

From the table above we see that odd triangular numbers are given as

1	3	15	21	45	55	91	105	153	171	231	253	325	351
1*1	1*3	3*5	3*7	5*9	5*11	7*13	7*15	9*17	9*19	11*21	11*23	13*25	13*27
t_1	t_2	t_5	t_6	t_9	t_{10}	t_{13}	t_{14}	t_{17}	t_{18}	t_{21}	t_{22}	t_{25}	t_{26}

$$\begin{cases} t_{2i-2}, & i \text{ is even} \\ & \text{and} \\ t_{2i-1}, & i \text{ is odd} \end{cases} \Rightarrow \begin{cases} t_{4k-2}, & for\ i = 2k\ , k \in \mathbb{Z}^+ \\ & \text{and} \\ t_{4k-3}, & for\ i = 2k-1, k \in \mathbb{Z}^+ \end{cases}$$

6	10	28	36	66	78	120	136	190	210	276	300	378	406
2*3	2*5	4*7	4*9	6*11	6*13	8*15	8*17	10*19	10*21	12*23	12*25	13*27	13*29
t_3	t_4	t_7	t_8	t_{11}	t_{12}	t_{15}	t_{16}	t_{19}	t_{20}	t_{23}	t_{24}	t_{27}	t_{28}

and in the table below the even triangular numbers has following subscripts,

$$\begin{cases} t_{2i}\ , & i \text{ is even} \\ & \text{and} \\ t_{2i+1}, & i \text{ is odd} \end{cases} \Rightarrow \begin{cases} t_{4k}\ , & for\ i = 2k\ , k \in \mathbb{Z}^+ \\ & \text{and} \\ t_{4k-1}\ , & for\ i = 2k-1, k \in \mathbb{Z}^+ \end{cases}$$

CONCLUSION AND REMARKS

The sum of two triangular numbers may be a triangular number. For instance the pairs (6 , 15) and (21, 45) are triangular number with 6 + 15 = $T_3 + T_5$ = 21 = T_6 and 21 + 45 = $T_6 + T_9 = T_1$ = 66 are again a triangular numbers. Moreover, if you see the double factorial,

5!! = 1. 3. 5 = = (1)(3.5) = $T_1.T_5$
9!! = 1.3.5.7.9 = (1)(3.7)(5.9) = $T_1.T_6.T_9$ and
13!! = 1.3.5.7.9.11.13 = (1)(7.13)(5.11)(3) (9) = $T_1\ .T_{13}\ T_{10}\ T_2^3$.

We ponder that the double factorial of odd integers can be expresses as a product of triangular numbers. Is it unique? Can we find a relationship between gamma functions, beta function and product of triangular numbers? Which even triangular numbers n has the form of $n = 2^{k-1}(2^k - 1)$ and is perfect. These are open problems we are working on and close to show these facts are true in our next paper.

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