Permutation Identities

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Abstract - *This study determined permutation identities using a model. This study was carried out by constructing an array of numbers in triangular form with entry permutation values called Permutation Triangle. The entries in this model were critically observed to determine permutation equations. Concerning the result, there are three existing permutation identities and three other equations were determined using the model. One of this is a general equation which involves an infinite number of permutation identities. The study recommends the use on the model for further critical investigation to find out other permutation identities.*

Keywords: *Statistics, Probability, Permutation Identities*

INTRODUCTION

The set S which elements are already ordered, rearranging its elements is a process of permutation [1]. The number of permutation of a set with n elements taken all at a time is n! [2]. Often, permutation used a tool to examine the statistical significance of an event under investigation [3]. Proving identities about combinations is bane for the students because its formula is more difficult than permutation. To prepare them on proving combination identities we need a simpler equation similar to a combination identity which is permutation identities. This study is focused on how to construct identities about permutations. The findings of this study will benefit the students and teachers because the result of this study will enhance their knowledge on permutation & combination and develop their skills on proving identities.

The following discussion presents the theories, concepts, generalizations, and ideas which aided and inspired the researcher to determine equations regarding permutation. Learning theories are diversified in scope, the learning theories which inspired the researchers to pursue this study is the motivation theory. To uplift the students' interest, the teachers should find a way to motivate the students' by giving interesting problems so that they will gain new knowledge [4]. Intrinsic motivation – where the learning and the acquiring knowledge, interest directly the student. The center of intrinsic motivation is curiosity, that means the desire to know much more [5].

Another theory which served as a guide and a model for the researchers is the Pascal Triangle theory. The Pascal triangle theory is an array of number in triangular form. Each entry in this array was determined using the combination formula. In a Pascal Triangle we can derive equations regarding combinations. The diagram (figure 1) below is the Pascal triangle:

```
                  1
               1     1
            1     2     2
         1     3     3     1
      1     4     6     4     1
   1     5    10    10     5     1
1     6    15    20    15     6     1
```

Figure 1. Pascal's Triangle

Pascal triangle was derived because of expansion of binomials with n exponents. From this Pascal triangle some of the identities involving combinations were derived using the method. Applying the same method maybe we can possibly determine some of the identities involving permutations. To do this, the researcher aimed to describe the :

1. Existing identities in permutation which can be utilized to construct a triangular form of numbers whose entries are permutation values,
2. Identities of permutations could be found out using a triangle form of numbers whose entries are permutation values (Permutation triangle).

METHODOLOGY

Permutation is an ordered arrangement of r on the n elements of the sets. It is denoted by P(n,r) the number of r permutations of an n – elements of set. If r=0,then P(n,r)=1. if r=n,then P(n,r)=n!.If r>n,then P(n,r)=0.If r<n,then P(n,r)=n!/(n-r)!.

Using the identity P(1,1)=1 and P(n,n-1)=P(n,n) we can form the first two rows of figure 2 and the last two entries in each row. To complete the figure without using the permutation formula we can use the following method. In constructing the third row the following are the procedures:

Step 1. Place three to the left and below of 2 this is the first entry of the third row,

Step 2. For the other entries, multiply the first entry of second row by three, place the first product to the right of the first entry and at the middle of the two entries in the second row.

To construct the other rows, the following are the procedures:

Let consider the fourth row

Step 1. Place four to the left and below of 3 this is the first entry of the fourth row.

Step 2. For the second entry of the fourth row multiply the first entry of third row by four, place the first product to the right of the first entry and at the middle of the first two entries in the third row.

Step 3. To obtain the third entry of the fourth row, multiply the second entry in the third row by four. Again, place the product to the right and in the middle of the last two entries of the third row align in the fourth row.

Step 4. Repeat this process until all entries in the third row is multiplied by four.

To repeat the process up to 7th row one could construct figure 1 below. The first entry in each row is increased by 1 from the first entry of the previous row.

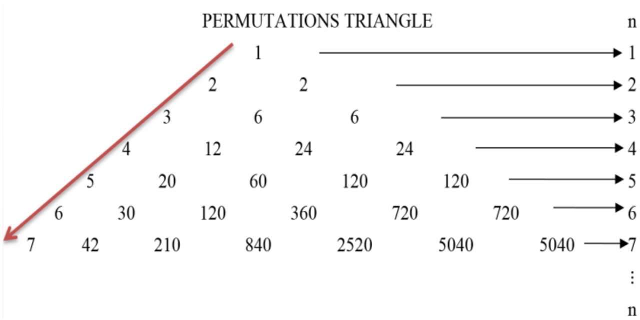

Figure 2. Permutation Triangle

We can generate this figure using the permutation formula or by multiplying the entries in the Pascal triangle by the corresponding number row position. The number from the left side pointed by the arrow considered a degenerate numbers, because they will generate other entries of the figure, by multiplying each of the other (i.e. 60=5x4x3,2520=7x6x5x4x3). This is another method to construct the figure 2. The model above gives us an indication that if we examine it critically it is possible to find out the identities in permutation.

RESULTS AND DISCUSSIONS

From figure (2), by critical investigation the following identities are constructed about permutation.

1. Figure (3), indicates that we can derive an identities about permutation, and this was the existing identity about permutation.

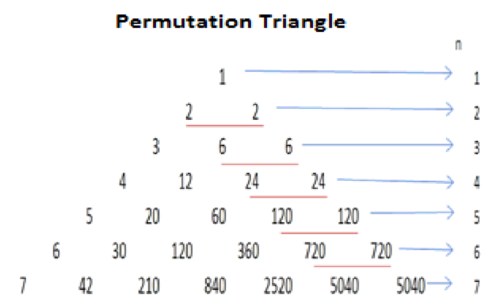

Figure 3

$P(n,n-1) = P(n,n)$

i. Verification:

a. Let $n = 5$

$P(n,n-1) = P(5,4) = 120$

$P(n,n) = P(5,5) = 120$

b. Let $n = 5$

$P(n,n-1) = P(7,6) = 5040$

$P(n,n) = P(7,7) = 5040$

$\therefore P(n,n-1) = P(n,n)$

ii. Proof:

$P(n,n-1) = \frac{n!}{(n-(n-1))!}$

$= \frac{\boldsymbol{n!}}{\boldsymbol{1!}} = n! = P(n,n)$

2. Figure (4) below, gives us also an indication to form other identities about permutation. The figure implies that $P(n,r) = nP(n-1,r-1)$.

Figure 4

i. Verification:

a. Let $n = 5\ and\ r = 3$

$P(n,r) = P(5,3) = 60$

$nP(n-1, r-1) = 5P(4,2) = 5(12) = 60$

b. Let $n = 7\ and\ r = 4$

$P(n,r) = P(7,4) = 840$

$nP(n-1, r-1) = 7P(6,3) = 7(120) = 840$

Therefore, $P(n,r) = nP(n-1, r-1)$.

ii. Proof:

$$nP(n-1, r-1) = n\left(\frac{(n-1)!}{[(n-1)-(r-1)]!}\right)$$
$$= \mathrm{n}\left(\frac{(n-1)!}{(n-r)!}\right)$$
$$= \left(\frac{n(n-1)!}{(n-r)!}\right)$$
$$= \frac{n!}{(n-r)!}$$
$$= P(n,r)$$ ★

3. Figure (5), indicates that we can derive an identities about permutation, again this was the existing identity about permutation.

$$P(n,r) = n(n-1)(n-2)\cdots(n-(r-1))$$

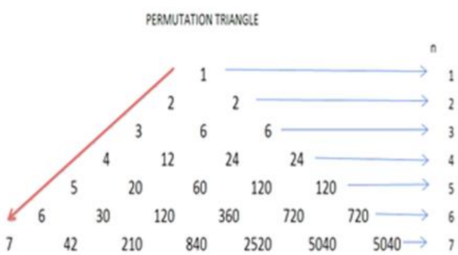

Figure 5

i. Verification:

a. Let $n = 5\ and\ r = 3$

$P(n,r) = P(5,3) = 60$

$n(n-1)(n-2) = 5(5-1)(5-2) = 5 \cdot 4 \cdot 3 = 60$

b. Let $n = 5\ and\ r = 4$

$P(n,r) = P(6,4) = 360$

$n(n-1)(n-2) = 6(6-1)(6-2)(6-3) = 6 \cdot 5 \cdot 4 \cdot 3 = 360$

Therefore, $P(n,r) = n(n-1)(n-2)\cdots(n-(r-1))$

ii. Proof

$$n(n-1)(n-2)\cdots\big(n-(r-1)\big) = P(n,1)P(n-1,1)P(n-2,1)\cdots P(n-(r-1),1)$$
$$= \frac{n!}{(n-1)!}\frac{(n-1)!}{((n-1)-1)!}\frac{(n-2)!}{((n-2)-1)!}\cdots\frac{(n-(r-1))!}{[(n-(r-1))-1]!}$$
$$= \frac{n!}{(n-1)!}\frac{(n-1)!}{(n-2)!}\frac{(n-2)!}{(n-3)!}\cdots\frac{(n-(r-1))!}{(n-r)!}$$
$$= \frac{n!}{(n-r)!}$$
$$= P(n,r)$$ ★

4. Figure (6), indicates that another pattern to generate an identities about permutation. Using this pattern we can generate an infinite number of identities of permutation. The six identities below are some of the identities created using this pattern.

P(n,n-1)=nP(n-1,n-1)

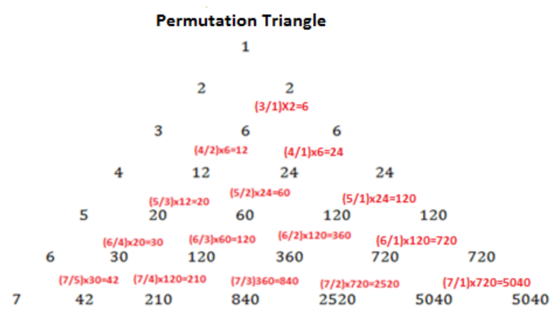

Figure 6

i. Verification:

a. Let $n = 5$

$P(n, n-1) = P(5,4) = 120$

$nP(n-1, n-1) = 5P(4,4) = 5(24) = 120$

b. Let $n = 7$

$P(n, n-1) = P(7,6) = 5040$

$nP(n-1, n-1) = 7P(6,6) = 7(720) = 5040$

Therefore, $P(n, n-1) = P(n-1, n-1)$

ii. Proof:

$$nP(n-1, n-1) = n \cdot \frac{(n-1)!}{[(n-1)-(n-1)]!}$$
$$= n \cdot \frac{(n-1)!}{[(n-1)-(n-1)]!}$$
$$= \frac{n(n-1)!}{(0)!}$$

$= n!$

$= P(n,n)$

Using transitive property and identity 1, therefore

$= P(n,n-1)$

5. $P(n,n-2) = \frac{n}{2}P(n-1,n-2)$

i. Verification:

a. Let $n = 5$

$P(n,n-2) = P(5,3) = 60$

$\frac{n}{2}P(n-1,n-2) = \frac{5}{2}P(4,3) = \frac{5}{2}(24) = 60$

b. Let $n = 7$

$P(n,n-2) = P(7,5) = 2{,}520$

$\frac{n}{2}P(n-1,n-2) = \frac{7}{2}P(6,4) = \frac{7}{2}(360) = 2{,}520$

Therefore, $P(n,n-2) = \frac{n}{2}P(n-1,n-2)$

ii. Proof:

$$\frac{n}{2}P(n-1,n-2) = \frac{n}{2}\cdot\frac{(n-1)!}{[(n-1)-(n-2)]!}$$
$$= \frac{n}{2}\cdot\frac{(n-1)!}{[(n-1)-(n-2)]!}$$
$$= \frac{n(n-1)!}{2(1)!}$$
$$= \frac{n!}{2!}$$
$$= \frac{n!}{(n-n+2)!}$$
$$= \frac{n!}{(n-(n-2))!}$$
$$= P(n,n-2)$$

6. $P(n,n-3) = \frac{n}{3}P(n-1,n-3)$

i. Verification:

a. Let $n = 5$

$P(n,n-3) = P(5,2) = 20$

$\frac{n}{3}P(n-1,n-3) = \frac{5}{3}P(4,2) = \frac{5}{3}(12) = 20$

b. Let $n = 7$

$P(n,n-3) = P(7,4) = 840$

$\frac{n}{3}P(n-1,n-3) = \frac{7}{3}P(6,4) = \frac{7}{3}(360) = 840$

Therefore, $P(n,n-3) = \frac{n}{3}P(n-1,n-3)$

ii. Proof:

$$\frac{n}{3}P(n-1,n-3) = \frac{n}{3}\cdot\frac{(n-1)!}{[(n-1)-(n-3)]!}$$
$$= \frac{n}{3}\cdot\frac{(n-1)!}{[(n-1)-(n-3)]!}$$
$$= \frac{n(n-1)!}{3(2)!}$$
$$= \frac{n!}{3!}$$
$$= \frac{n!}{(n-n+3)!}$$
$$= \frac{n!}{(n-(n-3))!}$$
$$= P(n,n-3)$$

7. $P(n,n-4) = \frac{n}{4}P(n-1,n-4)$

i. Verification:

a. Let $n = 5$

$P(n,n-4) = P(5,1) = 5$

$\frac{n}{4}P(n-1,n-4) = \frac{5}{4}P(4,1) = \frac{5}{4}(4) = 5$

b. Let $n = 7$

$P(n,n-4) = P(7,3) = 210$

$\frac{n}{4}P(n-1,n-4) = \frac{7}{4}P(6,3) = \frac{7}{4}(120) = 210$

Therefore, $P(n,n-4) = \frac{n}{4}P(n-1,n-4)$

ii. Proof

$$\frac{n}{4}P(n-1,n-4) = \frac{n}{4}\cdot\frac{(n-1)!}{[(n-1)-(n-4)]!}$$
$$= \frac{n}{4}\cdot\frac{(n-1)!}{[(n-1)-(n-4)]!}$$
$$= \frac{n(n-1)!}{4(3)!}$$
$$= \frac{n!}{4!}$$
$$= \frac{n!}{(n-n+4)!}$$
$$= \frac{n!}{(n-(n-4))!}$$
$$= P(n,n-4)$$

8. $P(n,2) = \frac{n}{n-2}P(n-1,2)$

i. Verification:

a. Let $n = 5$

$P(n,2) = P(5,2) = 20$

$\frac{n}{n-2}P(n-1,2) = \frac{5}{3}P(4,2) = \frac{5}{3}(12) = 20$

b. Let $n = 7$

$P(n,2) = P(7,2) = 42$

$\frac{n}{n-2}P(n-1,2) = \frac{7}{5}P(6,2) = \frac{7}{5}(30) = 42$

Therefore, $P(n,2) = \frac{n}{n-2}P(n-1,2)$

ii. Proof:

$$\frac{n}{n-2}P(n-1,2) = \frac{n}{n-2}\cdot\frac{(n-1)!}{[(n-1)-(n-4)]!}$$
$$= \frac{n}{n-2}\cdot\frac{(n-1)!}{[(n-1)-2]!}$$
$$= \frac{n(n-1)!}{(n-2)(n-3)!}$$
$$= \frac{n!}{(n-2)!}$$
$$= P(n,2)$$ ★

9. Identity 4 – 8 can be presented generally using the equations below,

$$P(n,n-\alpha) = \frac{n}{\alpha}P(n-1,n-\alpha) \ \ where, \alpha \in Z \ and \ \alpha < n.$$

i. Proof

$$\frac{n}{\alpha}P(n-1,n-\alpha) = \frac{n}{\alpha}\cdot\frac{(n-1)!}{[(n-1)-(n-\alpha)]!}$$
$$= \frac{n(n-1)!}{\alpha(\alpha-1)!}$$
$$= \frac{n!}{\alpha!}$$
$$= \frac{n!}{(n-n+\alpha)!}$$
$$= \frac{n!}{(n-(n-\alpha))!}$$
$$= P(n,n-\alpha)$$ ★

FINDINGS

Through the process of critical investigation using the permutation triangle, the following identities below were created, and was verified and proved. The first three were the existing identities about permutations which could be found using permutation triangle, the other identities were also found using the permutation triangle.

1. $P(n,n-1) = P(n,n)$
2. $P(n,r) = nP(n-1,r-1)$
3. $P(n,r) = n(n-1)(n-2)\cdots(n-(r-1))$
4. $P(n,2) = \frac{n}{n-2}P(n-1,2)$
5. $P(n,n-1) = nP(n-1,n-1)$
6. $P(n,n-2) = \frac{n}{2}P(n-1,n-2)$
7. $P(n,n-3) = \frac{n}{3}P(n-1,n-3)$
8. $P(n,n-4) = \frac{n}{4}P(n-1,n-4)$

⋮ ⋮ ⋮

$P(n,n-\alpha) = \frac{n}{\alpha}P(n-1,n-\alpha)$ $where, \alpha \in Z \ and \ \alpha < n$

CONCLUSION AND RECOMMENDATION

The triangular form of numbers whose entries are a permutation of values is one of the models which we can be used to find other identities in permutation. From the derived identities using this method was verified which are fairly consistent and accurate.

The derived identities proved that they were true, for further critical investigation about the model one can construct more identities about permutation. Therefore, it is recommended the use of the model for further critical investigation be further utilized for other permutation identities.

REFLECTIONS

The identities presented above were created through permutation triangle, these were some of the identities of the infinite number of identities one could give value for α in equation 9 that one can create using this method. For the expansion of this study, using the same method are there any other identities that one could create? What are the other methods that one can use to create permutation identities?

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