Recurrence Relations and Generating Functions of the Sequence of Sums of Corresponding Factorials and Triangular Numbers

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Abstract – *This study established some recurrence relations and exponential generating functions of the sequence of factoriangular numbers. A factoriangular number is defined as a sum of corresponding factorial and triangular number. The proofs utilize algebraic manipulations with some known results from calculus, particularly on power series and Maclaurin's series. The recurrence relations were found by manipulating the formula defining a factoringular number while the ascertained exponential generating functions were in the closed form.*

Keywords – *closed formula, factorial, factoriangular number, exponential generating function, recurrence relation, sequence, triangular number*

INTRODUCTION

The succession of numbers formed according to a fixed rule is called a *sequence* [1]. A function whose domain is the set $\{1, 2, ..., n, ...\}$ is called a *sequence function* and the numbers in the range are called *elements* of the sequence [2]. For example, $1, 4, 9, 16, 25, 36, ...$ is a sequence having the rule that the *n*th term is given by n^2, while $1, 3, 6, 10, 15, 21, ...$ is a sequence having the rule that the *n*th term is given by $n(n+1)/2$. The elements of the first example sequence are called square numbers while the elements of the second example sequence are called triangular numbers. If f is the function defining the sequence, then the sequence function is $f(n)=n^2$ for the first example and $f(n)=n(n+1)/2$ for the second example. A sequence can be denoted by the function notation $\{f(n)\}$ or by the subscript notation $\{a_n\}$. Thus, if $f(n)=T_n=n(n+1)/2$ then the sequence of triangular numbers may also be denoted by either $\{T_n\}$ or $\{n(n+1)/2\}$.

Given the sequence $a_0, a_1, a_2, a_3, ...$, there is also a systematic way of getting the *n*th term from the preceding terms $a_{n-1}, a_{n-2}, ...$. The rule for doing this is called *recurrence relation*, which provides a method for determining as many terms of the sequence as desired [3]. For instance, given a term of a triangular number $T_n = n(n+1)/2$, the succeeding terms in the sequence of triangular numbers are determined by the relation $T_{n+1} = T_n + (n+1)$.

In the study of sequences and recurrences, it is usually helpful to represent the sequence by power series such as

$$A(x)=\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...,$$

the *ordinary generating function* of the sequence, or

$$E(x)=\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + ...,$$

the *exponential generating function*. Generating functions provide a very efficient way of representing sequences [3]. Note that a *power series* is a generalization of a polynomial function [2].

When a recurrence for a sequence has been found, its generating function (GF) can be solved by using several techniques (see [3]). For example, the Fibonacci numbers $1, 1, 2, 3, 5, ...$ are generated by a relation $a_n = a_{n-1} + a_{n-2}$, from which it is not difficult to obtain a closed form GF,

$$\frac{1}{1-x-x^2},$$

as follows:

Let

$$A(x)=\sum_{n=0}^{\infty}a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

be the GF. Since $a_0 = 1$,

$$A(x) = 1 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

and since also $a_1 = 1$,

$$A(x) = 1 + x + a_2 x^2 + ... + a_n x^n + ...$$

Since $a_n = a_{n-1} + a_{n-2}$ it further follows that

$$A(x) = 1 + x + \sum_{n=2}^{\infty} a_n x^n$$

$$\Leftrightarrow \quad A(x) = 1 + x + \sum_{n=2}^{\infty}(a_{n-1} + a_{n-2})x^n$$

$$\Leftrightarrow \quad A(x) = 1 + x + \sum_{n=2}^{\infty} a_{n-1}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n$$

$$\Leftrightarrow \quad A(x) = 1 + x + (a_1x^2 + a_2x^3 + a_3x^4 + ...) + (a_0x^2 + a_1x^3 + a_2x^4 + ...)$$

$$\Leftrightarrow \quad A(x) = 1 + x + x(a_1x + a_2x^2 + a_3x^3 + ...) + x^2(a_0 + a_1x + a_2x^2 + ...)$$

$$\Leftrightarrow \quad A(x) = 1 + x + x(A(x) - 1) + x^2 A(x)$$

$$\Leftrightarrow \quad A(x) = 1 + x + xA(x) - x + x^2 A(x)$$

$$\Leftrightarrow \quad A(x) = 1 + xA(x) + x^2 A(x)$$

$$\Leftrightarrow \quad A(x) - xA(x) - x^2 A(x) = 1$$

$$\Leftrightarrow \quad (1 - x - x^2)A(x) = 1$$

$$\Leftrightarrow \quad A(x) = \frac{1}{1-x-x^2}.$$

A great deal had also been written about how generating functions can be used in mathematics (see [3] for some references).

OBJECTIVES OF THE STUDY

This study aimed to establish some recurrence relations and exponential generating functions of the relatively new sequence of numbers, the so-called *factoriangular numbers*. These numbers are found by adding corresponding factorials and triangular numbers, $n! + T_n$, and forming the sequence

$$2,\ 5,\ 12,\ 34,\ 135,\ 741,\ 5068,\ 40356,\ 362925,\ ...,$$

which appears as sequence A101292 in [4].

The name *factoriangular* was introduced in [5] as a contraction of the terms *factorial* and *triangular*, together with some preliminary results on the characteristics of such numbers including parity, compositeness, number and sum of positive divisors, abundancy and deficiency, Zeckendorf's decomposition, end digits and digital roots. In [6], factoriangular numbers were represented as runsums and as difference of two triangular numbers, and the politeness, as well as trapezoidal arrangements associated with factoriangular numbers were also studied. Further in [7], interesting results on the representations of factoriangular numbers as sum of two triangular numbers and as sum of two squares were presented.

METHODS

This study employs basic and expository research method. It utilizes mathematical exposition and exploration, along with ingenuity and inventiveness to arrive at a proof. It basically uses algebraic methods and manipulations but utilizing also known results from calculus, particularly on power series and Maclaurin's series.

RESULTS AND DISCUSSION

A factoriangular number was defined in [5] as:

Definition. The nth factoriangular number is given by the formula $Ft_n = n! + T_n$, *where* $n! = 1\cdot 2\cdot 3\cdots n$ *and* $T_n = 1 + 2 + 3 + ... + n = n(n+1)/2$.

To find a relationship between two consecutive factoriangular numbers, Ft_n and Ft_{n+1}, the equations defining them can be manipulated to have $n!$ on one side of the equation as follows:

$$Ft_n = n! + T_n$$

$$\Leftrightarrow \quad Ft_n - T_n = n!$$

$$\Leftrightarrow \quad Ft_n - \frac{n(n+1)}{2} = n!,$$

and

$$Ft_{n+1} = (n+1)! + T_{n+1}$$

$$\Leftrightarrow \quad Ft_{n+1} - T_{n+1} = (n+1)!$$

$$\Leftrightarrow \quad Ft_{n+1} - \frac{(n+1)(n+2)}{2} = (n+1)n!$$

$$\Leftrightarrow \quad \frac{Ft_{n+1}}{n+1} - \frac{n+2}{2} = n!.$$

Equating the two expressions that are both equal to $n!$ and then simplifying,

$$\frac{Ft_{n+1}}{n+1} - \frac{n+2}{2} = Ft_n - \frac{n(n+1)}{2}$$

$$\Leftrightarrow \quad \frac{Ft_{n+1}}{n+1} = Ft_n - \frac{n^2+n}{2} + \frac{n+2}{2}$$

$$\Leftrightarrow \quad Ft_{n+1} = (n+1)\left(Ft_n - \frac{n^2-2}{2} \right).$$

The above serves as proof for:

Theorem 1. For n ≥ 1, the factoriangular numbers follow the recurrence relation

$$Ft_{n+1} = (n+1)\left(Ft_n - \frac{n^2-2}{2} \right).$$

In an analogous way, another recurrence relation involving Ft_n and Ft_{n-1} can be established. Consider the following:

$$Ft_n = n! \ + \ T_n$$

$$\Leftrightarrow \quad Ft_n - T_n = n!$$

$$\Leftrightarrow \quad Ft_n - \frac{n(n+1)}{2} = n(n-1)!$$

$$\Leftrightarrow \quad \frac{Ft_n}{n} - \frac{n+1}{2} = (n-1)!,$$

and

$$Ft_{n-1} = (n-1)! \ + \ T_{n-1}$$

$$\Leftrightarrow \quad Ft_{n-1} - T_{n-1} = (n-1)!$$

$$\Leftrightarrow \quad Ft_{n-1} - \frac{n(n-1)}{2} = (n-1)!.$$

Equating these two and simplifying,

$$\frac{Ft_n}{n} - \frac{n+1}{2} = Ft_{n-1} - \frac{n(n-1)}{2}$$

$$\Leftrightarrow \quad Ft_n = n\left(Ft_{n-1} - \frac{n^2-n}{2} + \frac{n+1}{2} \right)$$

$$\Leftrightarrow \quad Ft_n = n\left(Ft_{n-1} - \frac{n^2-2n-1}{2} \right).$$

The above is the proof of a theorem that is stated formally as:

Theorem 2. For n ≥ 2, the recurrence relation

$$Ft_n = n\left(Ft_{n-1} - \frac{n^2-2n-1}{2} \right)$$

holds true for factoriangular numbers.

Notice the restriction $n \geq 2$. If $n = 1$ then $Ft_{n-1} = Ft_0$, which is excluded in the definition of factoriangular number. Nevertheless, if $Ft_0 = 0! + T_0 = 1 + 0 = 1$ is to be included, Theorem 2 still holds true. Similarly, if Ft_0 is to be included, Theorem 1 still holds true for $n \geq 0$. But n cannot be equal to zero for Theorem 2 since Ft_{-1} is not defined.

With the formula defining a factoriangular number, the first factoriangular number is computed as

$$Ft_1 = 1! + \frac{1(1+1)}{2} = 2.$$

Then, subsequent factoriangular numbers may be computed by using a recurrence relation. For instance, given that $Ft_1 = 2$, then using the relation in Theorem 1, the next factoriangular numbers are computed as follows:

$$Ft_2 = 2\left(2 - \frac{1^2-2}{2} \right) = 5,$$

$$Ft_3 = 3\left(5 - \frac{2^2-2}{2} \right) = 12,$$

$$Ft_4 = 4\left(12 - \frac{3^2-2}{2} \right) = 34,$$

and so on.

Using the recurrence relation, all factoriangular numbers can be generated. The sequence of factoriangular numbers is given by

$$\{Ft_n\} = 2, 5, 12, 34, 135, 741, 5068, 40356, 362925, ...$$

for $n \geq 1$. This sequence has an exponential generating function (EGF) stated in the following theorem:

Therorem 3. The exponential generating function of Ft_n for n ≥ 1 is given by the closed formula

$$E(x) = \frac{2 + (2 - 5x^2 + 2x^3 + x^4)e^x}{2(1-x)^2}$$

for $-1 < x < 1$.

Proof:

For the sequence $2, 5, 12, 34, 135, ...$, the EGF is

$$E(x) = 2 + 5x + 12\frac{x^2}{2!} + 34\frac{x^3}{3!} + 135\frac{x^4}{4!} + ...$$

and each term can be decomposed into having

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$$E(x) = 1+2x+6\frac{x^2}{2!}+24\frac{x^3}{3!}+120\frac{x^4}{4!}+\dots$$
$$+1+3x+6\frac{x^2}{2!}+10\frac{x^3}{3!}+15\frac{x^4}{4!}+\dots.$$

The first set of summands can be simplified while the second set can be factored as follows:

$$E(x) = 1+2x+3x^2+4x^3+5x^4+\dots$$
$$+\left(1+2x+\frac{x^2}{2}\right)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right).$$

Since $\frac{1}{1-x}=1+x+x^2+x^3+\dots$, for $-1<x<1$, which when both sides are differentiated gives $\frac{1}{(1-x)^2}=1+2x+3x^2+\dots$, for $-1<x<1$, and since $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$, for all values of x (see [1] and [2]), it follows that

$$E(x)=\frac{1}{(1-x)^2}+\left(1+2x+\frac{x^2}{2}\right)e^x$$
$$\Leftrightarrow \quad E(x)=\frac{1}{(1-x)^2}+\left(\frac{2+4x+x^2}{2}\right)e^x$$
$$\Leftrightarrow \quad E(x)=\frac{2+(1-x)^2(2+4x+x^2)e^x}{2(1-x)^2}$$
$$\Leftrightarrow \quad E(x)=\frac{2+(2-5x^2+2x^3+x^4)e^x}{2(1-x)^2}$$

and the proof is completed.

If the factoriangular number for n = 0 is included, then the sequence of factoriangular numbers for $n \geq 0$ is given by

$\{Ft_n\}=1,2,5,12,34,135,741,5068,40356,362925,\dots.$

In this case, the sequence has the following EGF:

Theorem 4. The exponential generating function of Ft_n for $n \geq 0$ is given by the closed formula

$$E(x)=\frac{2+(2x-x^2-x^3)e^x}{2(1-x)}$$

for $-1<x<1$.

Proof:

For the sequence $1, 2, 5, 12, 34, \dots$, the *egf* is

$$E(x)=1+2x+5\frac{x^2}{2!}+12\frac{x^3}{3!}+34\frac{x^4}{4!}+\dots.$$

This can also be decomposed into two sets of summands to have

$$E(x) = 1+x+2\frac{x^2}{2!}+6\frac{x^3}{3!}+24\frac{x^4}{4!}+\dots$$
$$+ \quad x+3\frac{x^2}{2!}+6\frac{x^3}{3!}+10\frac{x^4}{4!}+\dots$$

and simplifying the first set while factoring the second set results to

$$E(x) = 1+x+x^2+x^3+x^4+\dots$$
$$+\left(x+\frac{x^2}{2}\right)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right).$$

Hence,

$$E(x)=\frac{1}{(1-x)}+\left(x+\frac{x^2}{2}\right)e^x$$
$$\Leftrightarrow \quad E(x)=\frac{1}{1-x}+\left(\frac{2x+x^2}{2}\right)e^x$$
$$\Leftrightarrow \quad E(x)=\frac{2+(2x-x^2-x^3)e^x}{2(1-x)},$$

which completes the proof.

CONCLUSIONS

A factoriangular number is defined as the sum of corresponding factorial and triangular number and is given by $n! + T_n$, where $T_n=n(n+1)/2$. Succession of these numbers forms the sequence of factoriangular numbers:

$2, 5, 12, 34, 135, 741, 5068, 40356, 362925, \dots.$

These factoriangular numbers follow the recurrence relations:

$$Ft_{n+1}=(n+1)\left(Ft_n-\frac{n^2-2}{2}\right), \text{ for } n\geq 1, \text{ and}$$
$$Ft_n=n\left(Ft_{n-1}-\frac{n^2-2n-1}{2}\right), \text{ for } n\geq 2.$$

The exponential generating function of the sequence of factoriangular numbers is given by the formula:

$$E(x)=\frac{2+(2-5x^2+2x^3+x^4)e^x}{2(1-x)^2}$$

for $n\geq 1$ and $-1<x<1$. If the sequence of factoriangular numbers is considered to begin with

$n=0$, then the sequence becomes $1, 2, 5, 12, 34, 135, 741, 5068, 40356, 362925, \ldots$ and the exponential generating function is

$$E(x)=\frac{2+(2x-x^2-x^3)e^x}{2(1-x)}$$

for $n \geq 0$ and $-1<x<1$.

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