Sums of Two Triangulars and of Two Squares Associated with Sum of Corresponding Factorial and Triangular Number

Asia Pacific Journal of Multidisciplinary Research
Vol. 3 No. 4, 28-36
November 2015 Part III
P-ISSN 2350-7756
E-ISSN 2350-8442
www.apjmr.com

Romer C. Castillo (M.Sc.)
Batangas State University, Batangas City, Philippines
romercastillo@rocketmail.com

Date Received: September 2, 2015; Date Revised: October 5, 2015

Abstract – *Factoriangular numbers resulted from adding corresponding factorials and triangular numbers. If Ft_n is the nth factoriangular number, n! is the factorial of n and T_n is the nth triangular number, then $Ft_n = n! + T_n$. In this study, interesting results on the representations of factoriangular number as sum of two triangular numbers and as sum of two squares are presented.*

Keywords – *factoriangular number, factorial, sum of two triangular numbers, sum of two squares, theory of numbers*

INTRODUCTION

J. Hadamard once wrote that a sense of beauty is almost the only useful drive for discovery in mathematics [1]. A. De Morgan once quoted: *the moving power of mathematical invention is not reasoning but imagination* [2]. And it is in number theory, one of the oldest fields in mathematics that most of the greatest mathematicians in history have tried their hand [3] paving the way for mathematical experimentations, explorations and discoveries. As C. F. Gauss once said, *mathematics is the queen of the sciences and the theory of numbers is the queen of mathematics* [4].

As far back as ancient Greece, mathematicians were doing number theory. Although it is probable that the ancient Greek mathematicians were indebted to the Babylonians and Egyptians on the properties of natural numbers, the origin of number theory are credited to Pythagoras and the Pythagoreans [5]. An important subset of natural numbers is the set of polygonal numbers, a term introduced by Heysicles to refer to numbers that are triangular, square and so forth. Pythagoreans illustrated polygonal numbers as arrangements of points in regular geometric patterns. Triangular numbers represent the number of points used to portray equilateral triangular patterns and square numbers represent square array of points [6].

Relating triangular and square numbers, Nicomachus stated that the sum of any two consecutive triangular numbers is a square number. In addition, Plutarch found that eight times a triangular number plus one equals a square [6]. From then on, many other important results on polygonal numbers had been established, one of the greatest of which is Fermat's polygonal number theorem which states that every positive integer is a sum of at most three triangular numbers, four square numbers, five pentagonal numbers, and *k* *k*-gonal numbers. Gauss proved the triangular case [7]; Euler left important results on Fermat's theorem and these were used by Lagrange to prove the square case, which was also proven independently by Jacobi [8]; and the full proof of Fermat's theorem was shown by Cauchy [9].

There are also several recent studies on the sums of triangular numbers and of squares. Farkas used the theory of theta functions to discover formulas for the number of representations of a natural number as sum of three triangular numbers and as sum of three squares and reproved that every natural number can be written as sum of two triangular numbers plus one square and as sum of two squares plus one triangular number [10]. By means of *q*-series, Sun proved that any natural number is a sum of an even square and two triangular numbers and that each natural number is a sum of a triangular number plus $x^2 + y^2$ for some $x, y \in \mathbb{Z}$ with $x \not\equiv y \pmod 2$ or $x = y > 0$ [11]. He also presented some conjectures on the mixed sums of squares and triangular numbers that later became the subject of the papers of Guo, Pan and Sun [12], Oh and Sun [13], and Kane [9].

Square triangular numbers or numbers that are both square and triangular are also well-studied. A square triangular number can be written as m^2 for some m and as $n(n+1)/2$ for some n and hence, is given by the equation $m^2 = n(n+1)/2$, which is a Diophantine equation for which integer solutions are stipulated [14]. These numbers are related to balancing numbers introduced by Behera and Panda [15] and balancing numbers are closely associated with cobalancing numbers [16]. Some properties of square triangular numbers and balancing and numbers were studied by Keskin and Karaatli [17].

Triangular numbers are also related to factorials and in fact, these are the most well-known analogs in number theory. While triangular number is the sum of the first *n* natural numbers, factorial is their product. The relationship between factorials and triangular numbers is also evident in the identity $(2n)! = 2^n \prod_{i=1}^{n} T_{2i-1}$, where $n!$ is the factorial of n and T_n is the *n*th triangular number [18].

Much had been studied about square numbers, triangular numbers and factorials, as well as sums of triangular numbers and sums of square numbers. The present work is an innovative study about the sum of corresponding factorials and triangular numbers and some associated sums of two triangular numbers and sums of two squares.

METHODS

This study is a discipline-based scholarship of discovery that utilizes experimental mathematics and expositions and adheres to scientific approach. Enduring and time-consuming trial-and-error methods pave the way to rigorous proofs of the theorems presented. The elliptic curve method (see [19]) was used in determining the prime factors of some relatively large numbers included in this study.

RESULTS AND DISCUSSION

In this study, the sum of a factorial and its corresponding triangular number is called *factoriangular number*. The following are the notations to be utilized in defining a factoriangular number: *n* for a natural number, *n*! for factorial of a natural number *n*, T_n for *nth* triangular number, and Ft_n for *nth* factoriangular number. The definition of factoriangular number is now given below:

Definition. The nth factoriangular number is given by the formula $Ft_n = n! + T_n$, *where* $n! = 1 \cdot 2 \cdot 3 \cdots n$ *and* $T_n = 1+2+3+...+n = n(n+1)/2$.

A factoriangular number written as sum of two triangular numbers is generally denoted by $Ft_n = T_x + T_y$. But this can take four different expressions depending upon the values of the positive integers *n*, *x*, and *y*, as follows:

Case 1: If $x = y = n$, then $Ft_n = T_x + T_y$ becomes $Ft_n = T_n + T_n = 2T_n$.

Case 2: If $x = y \neq n$, then $Ft_n = T_x + T_y$ becomes $Ft_n = T_x + T_x = 2T_x$.

Case 3: If $x \neq y$ and $y = n$, then $Ft_n = T_x + T_y$ becomes $Ft_n = T_x + T_n$. (Note that the expression is similar if $x \neq y$ and $x = n$, that is $Ft_n = T_n + T_y$.)

Case 4: If $x \neq y \neq n$, then $Ft_n = T_x + T_y$, the general case.

After examining all possibilities for Case 1, the following theorem has been established:

Theorem 1. For $n \geq 1$, Ft_n is the nth factoriangular number, and T_n is the nth triangular number, only the two pairs: $Ft_1 = 2$ and $T_1 = 1$, and $Ft_3 = 12$ and $T_3 = 6$, satisfy the relation $Ft_n = 2T_n$.

Proof:

Since

$$Ft_n = 2T_n$$

$$\Leftrightarrow \quad n! + T_n = 2T_n$$

$$\Leftrightarrow \quad n! = T_n,$$

proving the theorem is just the same as finding integer solutions to the equation

$$n! = \frac{n(n+1)}{2}.$$

Hence, for $n = 1$,

$$1! = \frac{1(1+1)}{2} \Leftrightarrow 1 = 1;$$

for $n = 2$,

$$2! \neq \frac{2(2+1)}{2} \Leftrightarrow 2 \neq 3;$$

and for $n = 3$,

$$3! = \frac{3(3+1)}{2} \Leftrightarrow 6 = 6.$$

Thus, 1 and 3 are solutions but 2 is not. It can be shown that there is no other solution by simply showing that for $n \geq 4$,

$$n! > \frac{n(n+1)}{2}.$$

Notice that for $n \geq 4$, $n!$ is a product of at least three integers greater than 1. For instance, 4! = 4(3)(2), 5! = 5(4)(3)(2), and so on. Further, as n gets larger, the number of factors of $n!$ increases and its value also gets larger. That is, for $n \geq 4$,

$$n! = n(n-1)(n-2)(n-3)! \geq n(n-1)(n-2).$$

Hence, there is a need to show that

$$n(n-1)(n-2) > \frac{n(n+1)}{2}$$

is always true for $n \geq 4$ and the proof is as follows:

$$n(n-1)(n-2) > \frac{n(n+1)}{2}$$

$$\Leftrightarrow \quad 2(n-1)(n-2) > n+1$$

$$\Leftrightarrow \quad 2n^2 - 6n + 4 > n+1$$

$$\Leftrightarrow \quad 2n^2 - 7n > -3$$

$$\Leftrightarrow \quad n^2 - \frac{7}{2}n > -\frac{3}{2}$$

$$\Leftrightarrow \quad n^2 - \frac{7}{2}n + \left(\frac{7}{4}\right)^2 > -\frac{3}{2} + \frac{49}{16}$$

$$\Leftrightarrow \quad \left(n - \frac{7}{4}\right)^2 > \frac{25}{16}$$

$$\Leftrightarrow \quad n - \frac{7}{4} > \frac{5}{4}$$

$$\Leftrightarrow \quad n > 3.$$

Clearly, $n > 3$ is always true for $n \geq 4$. Therefore, $n! \geq n(n-1)(n-2) > n(n+1)/2$, for $n \geq 4$, and the only solutions to $n! = n(n+1)/2$ are $n = 1$ and $n = 3$, which imply that $Ft_n = 2T_n$ is true only for $(Ft_n, T_n) = (Ft_1, T_1), (Ft_3, T_3)$.

In connection to Case 2, the following theorem is hereby established:

Theorem 2. For n, $x \geq 1$, Ft_n is the nth fatoriangular number and T_x is the xth triangular number, $Ft_n = 2T_x$ if and only if $4Ft_n + 1$ is a square.

Proof:

Applying the definitions of factoriangular number and triangular number,

$$Ft_n = 2T_x$$

$$\Leftrightarrow \quad n! + \frac{n(n+1)}{2} = 2\left[\frac{x(x+1)}{2}\right]$$

$$\Leftrightarrow \quad \frac{2n! + n(n+1)}{2} = x^2 + x$$

$$\Leftrightarrow \quad x^2 + x + \frac{1}{4} = \frac{2n! + n(n+1)}{2} + \frac{1}{4}$$

$$\Leftrightarrow \quad \left(x + \frac{1}{2}\right)^2 = \frac{2[2n! + n(n+1)] + 1}{4}$$

$$\Leftrightarrow \quad \left(x + \frac{1}{2}\right)^2 = \frac{2n[2(n-1)! + (n+1)] + 1}{4}$$

$$\Leftrightarrow \quad x + \frac{1}{2} = \frac{\sqrt{2n[2(n-1)! + n + 1] + 1}}{2}$$

$$\Leftrightarrow \quad x = \frac{\sqrt{2n[2(n-1)! + n + 1] + 1} - 1}{2}.$$

Since

$$\frac{2Ft_n}{n} = \frac{2\left[n! + \frac{n(n+1)}{2}\right]}{n}$$

$$\Leftrightarrow \quad \frac{2Ft_n}{n} = \frac{2n(n-1)! + n(n+1)}{n}$$

$$\Leftrightarrow \quad \frac{2Ft_n}{n} = 2(n-1)! + n + 1,$$

it follows that,

$$x = \frac{\sqrt{2n\left(\frac{2Ft_n}{n}\right) + 1} - 1}{2}$$

$$\Leftrightarrow \quad x = \frac{\sqrt{4Ft_n + 1} - 1}{2},$$

which implies that x is a positive integer and $Ft_n = 2T_x$ if and only if $4Ft_n + 1$ is a square.

Looking for some solutions: for $n = 1$, $4Ft_1 + 1 = 4(2) + 1 = 9$ and $x = (\sqrt{9} - 1)/2 = 1$; for $n = 2$, $4Ft_2 + 1 = 4(5) + 1 = 21$ and there is no positive integer solution for x; and for $n = 3$, $4Ft_3 + 1 = 4(12) + 1 = 49$ and $x = (\sqrt{49} - 1)/2 = 3$. The solutions $(n, x) = (1, 1), (3, 3)$ for $Ft_n = 2T_x$ are the same solutions $n = 1, 3$ for $Ft_n = 2T_n$ of Case 1. Thus, if there is a condition that $n \neq x$ for $Ft_n = 2T_x$, then (1, 1) and (3, 3) are not solutions.

Considering the condition that $4Ft_n + 1$ should be a square to have $Ft_n = 2T_x$, the prime factors of $4Ft_n + 1$ for $n \leq 20$ are presented in Table 1. Notice there that only $4Ft_1 + 1 = 9$ and $4Ft_3 + 1 = 49$ can be expressed as square. A conjecture for this is presented as follows:

Conjecture 1. For n, $x \geq 1$ and $n \neq x$, there is no pair of factoriangular number, Ft_n, and triangular number, T_x, that satisfy the relation $Ft_n = 2T_x$.

Table 1. Prime Factors of $4Ft_n + 1$ for $n \leq 20$

n	Ft_n	$4Ft_n + 1$	Prime Factors
1	2	9	3^2
2	5	21	3 • 7
3	12	49	7^2
4	34	137	137
5	135	541	541
6	741	2965	5 • 593
7	5068	20273	11 • 19 • 97
8	40356	161425	5^2 • 11 • 587
9	362925	1451701	1187 • 1223
10	3628855	14515421	109 • 133169
11	39916866	159667465	5 • 163 • 409 • 479
12	479001678	1916006713	29 • 4759 • 13883
13	6227020891	24908083565	5 • 23 • 216592031
14	87178291305	348713165221	79 • 24967 • 176797
15	1307674368120	5230697472481	13 • 43 • 9357240559
16	20922789888136	83691159552545	5 • 449 • 503 • 74113147
17	355687428096153	1422749712384613	911 • 3967 • 19471 • 20219
18	6402373705728171	25609494822912685	5 • 6203 • 142757 • 5784047
19	121645100408832190	486580401635328761	1693 • 22697 • 12662783941
20	2432902008176640210	9731608032706560841	109752217 • 88668897073

Considering both cases 1 and 2, a similar conjecture is given as follows:

Conjecture 2. $Ft_1 = 2$ and $Ft_3 = 12$ are the only factoriangular numbers that is twice a triangular number.
With the proof of Theorem 1, if the proof for Conjecture 1 is found, Conjecture 2 is, as well, proven.

Case 3 is also somewhat similar to Case 1 where solving $Ft_n = T_x + T_n$ will be reduced to finding a factorial that is also triangular. Relative to this, another theorem is hereby established:

Theorem 3. For n, $x \geq 1$, Ft_n is the nth triangular number, and T_i is the ith triangular number, $Ft_n = T_x + T_n$ if and only if $8n! + 1$ is a square.

Proof:

Applying the definitions of factoriangular number and triangular number,

$$Ft_n = T_x + T_n$$

$$\Leftrightarrow \quad n! + \frac{n(n+1)}{2} = \frac{x(x+1)}{2} + \frac{n(n+1)}{2}$$

$$\Leftrightarrow \quad n! = \frac{x(x+1)}{2}$$

$$\Leftrightarrow \quad 2n! = x^2 + x$$

$$\Leftrightarrow \quad x^2 + x + \frac{1}{4} = 2n! + \frac{1}{4}$$

$$\Leftrightarrow \quad \left(x + \frac{1}{2}\right)^2 = \frac{8n!+1}{4}$$

$$\Leftrightarrow \qquad x + \frac{1}{2} = \frac{\sqrt{8n!+1}}{2}$$

$$\Leftrightarrow \qquad x = \frac{\sqrt{8n!+1}-1}{2},$$

which implies that x is a positive integer and $Ft_n = T_x + T_n$ if and only if $8n! + 1$ is a square. Three solutions are given here: for $n = 1$, $8(1!)+1=9$ and $x=(\sqrt{9}-1)/2=1$; for $n = 2$, $8(3!)+1=49$ and $x=(\sqrt{49}-1)/2=3$; and for $n = 3$, $8(5!)+1=961$ and $x=(\sqrt{961}-1)/2=15$.

Notice again that the solutions $(n, x) = (1, 1), (3, 3)$ are the same solutions found in Case 1 for $Ft_n = 2T_n$. These, however, are not to be considered as solutions for $Ft_n = T_x + T_n$ if there is a condition that $n \neq x$ and hence, only $(n, x) = (5, 15)$ is a solution. The prime factors of $8n! + 1$ for $n \leq 20$ are presented in Table 2.

Table 2. Prime Factors of $8n! + 1$ for $n \leq 20$

n	**8*n*! + 1**	**Prime Factors**
1	9	3^2
2	17	17
3	49	7^2
4	193	193
5	961	31^2
6	5761	7 · 823
7	40321	61 · 661
8	322561	47 · 6863
9	2903041	2903041
10	29030401	29030401
11	319334401	319334401
12	3832012801	3832012801
13	49816166401	307 · 162267643
14	697426329601	67 · 73 · 311 · 458501
15	10461394944001	10461394944001
16	167382319104001	47 · 521 · 6835558423
17	2845499424768001	863 · 5147 · 640609741
18	51218989645824001	93561439 · 547436959
19	973160803270656001	15375593 · 63292570457
20	19463216065413120001	4919 · 3956742440620679

Notice that only three $(8n! + 1)$ from the list in Table 2 can be represented as a square. In addition, by examining the end digits of both factorial and triangular numbers it can be deduced that $Ft_n = T_x + T_n$ or $n! = x(x+1)/2$ is true only if, x belongs to any of the sequences {15, 35, 55, 75, 95, 115, 135, …}, {20, 40, 60, 80, 100, 120, 140, …} or {24, 44, 64, 84, 104, 124, 144, …}. Note that $n = 5$ and $x = 15$ is the first pair of integer solution and it is also believed that this is the only solution, considering that as n gets larger, the trail of zeros in the end digits of $n!$ increases and makes it more improbable to have a triangular number equal to this factorial. Hence, a conjecture is given as follows:

Conjecture 3. For $n, x \geq 1$ and $n \neq x$, Ft_n is the nth factoriangular number and T_i is the ith triangular number, the only solution for $Ft_n = T_x + T_n$ is $(n, x) = (5, 15)$.

For the last case, the general case $Ft_n = T_x + T_y$, the following theorem established:

Theorem 4. For $n, x \geq 1$, Ft_n is the nth factoriangular number and T_i is ith triangular number, $Ft_n = T_x + T_y$ if and only if $8Ft_n + 2$ is a sum of two squares.

Proof:

$$Ft_n = T_x + T_y$$

$$\Leftrightarrow \quad Ft_n = \frac{x(x+1)}{2} + \frac{y(y+1)}{2}$$

$$\Leftrightarrow \quad 2Ft_n = (x^2 + x) + (y^2 + y)$$

$$\Leftrightarrow \quad 2Ft_n + \frac{1}{2} = \left(x^2 + x + \frac{1}{4}\right) + \left(y^2 + y + \frac{1}{4}\right)$$

$$\Leftrightarrow \quad 8Ft_n + 2 = (4x^2 + 4x + 1) + (4y^2 + 4y + 1)$$

$$\Leftrightarrow \quad 8Ft_n + 2 = (2x+1)^2 + (2y^2+1)^2,$$

which implies that for $Ft_n = T_x + T_y$, $8Ft_n + 2$ must be a sum of two squares.

It was already proven that any positive integer *n* can be represented by a sum of two squares if and only if it contains no prime factor of the form 4*k* + 3, for integer $k \geq 0$, raised to an odd power [20]. In Table 3, the prime factors of $8Ft_n + 2$ for $n \leq 20$ are presented.

Table 3. Prime Factors of $8Ft_n + 2$ for $n \leq 20$

n	Ft_n	$8Ft_n + 2$	Prime Factors
1	2	18	$2 \cdot 3^2$
2	5	42	$2 \cdot 3 \cdot 7$
3	12	98	$2 \cdot 7^2$
4	34	274	$2 \cdot 137$
5	135	1082	$2 \cdot 541$
6	741	5930	$2 \cdot 5 \cdot 593$
7	5068	40546	$2 \cdot 11 \cdot 19 \cdot 97$
8	40356	322850	$2 \cdot 5^2 \cdot 11 \cdot 587$
9	362925	2903402	$2 \cdot 1187 \cdot 1223$
10	3628855	29030842	$2 \cdot 109 \cdot 133169$
11	39916866	319334930	$2 \cdot 5 \cdot 163 \cdot 409 \cdot 479$
12	479001678	3832013426	$2 \cdot 29 \cdot 4759 \cdot 13883$
13	6227020891	49816167130	$2 \cdot 5 \cdot 23 \cdot 216592031$
14	87178291305	697426330442	$2 \cdot 79 \cdot 24967 \cdot 176797$
15	1307674368120	10461394944962	$2 \cdot 13 \cdot 43 \cdot 9357240559$
16	20922789888136	167382319105090	$2 \cdot 5 \cdot 449 \cdot 503 \cdot 74113147$
17	355687428096153	2845499424769226	$2 \cdot 911 \cdot 3967 \cdot 19471 \cdot 20219$
18	6402373705728171	51218989645825370	$2 \cdot 5 \cdot 6203 \cdot 142757 \cdot 5784047$
19	121645100408832190	973160803270657522	$2 \cdot 1693 \cdot 22697 \cdot 12662783941$
20	2432902008176640210	19463216065413121682	$2 \cdot 109752217 \cdot 88668897073$

In the examination of the prime factors of $8Ft_n + 2$, it was found that most have prime factors of the form 4*k* + 3, integer $k \geq 0$, occurring in odd multiplicity and hence, cannot be expressed as sum of two squares. The few $8Ft_n + 2$ and its representation as sum of two squares and the corresponding Ft_n and its representation as sum of two triangular numbers are given in Table 4.

Table 4. Representation of $8Ft_n + 2$ as Sum of Two Squares and of Ft_n as Sum of Two Triangular Numbers

n	$8Ft_n + 2$	Sum of Two Squares	Ft_n	Sum of Two Triangular Numbers
1	18	$3^2 + 3^2$	2	$T_1 + T_1$
3	98	$7^2 + 7^2$	12	$T_3 + T_3$
4	274	$7^2 + 15^2$	34	$T_3 + T_7$
5	1082	$11^2 + 31^2$	135	$T_5 + T_{15}$
6	5930	$47^2 + 61^2$	741	$T_{23} + T_{30}$
10	29030842	$539^2 + 5361^2$	3628855	$T_{269} + T_{2680}$
19	973160803270657522	$452955241^2 + 876351729^2$	121645100408832190	$T_{226477620} + T_{438175864}$
20	19463216065413121682	$1493461361^2 + 4151239481^2$	2432902008176640210	$T_{746730680} + T_{2075619740}$

Since:

$$Ft_n = T_x + T_y \Leftrightarrow 8Ft_n + 2 = (2x+1)^2 + (2y+1)^2,$$

if the squares are found, then the triangular numbers can be easily determined. For example, if $8Ft_4 + 2 = 274 = 7^2 + 15^2$, then $8Ft_4 + 2 = 274 \Leftrightarrow Ft_4 = 34$, $2x+1 = 7 \Leftrightarrow x = 3$ and $2y+1 = 15 \Leftrightarrow y = 7$. Hence, $Ft_4 = T_3 + T_7$.

Further, if $8Ft_n + 2$ can be represented by two different sums of two squares, then Ft_n can also be represented by two different sums of two triangular numbers. For instance,
$8Ft_{10} + 2 = 29030842 = 539^2 + 5361^2 \Rightarrow Ft_{10} = T_{269} + T_{2680}$ and
$8Ft_{10} + 2 = 29030842 = 3401^2 + 4179^2 \Rightarrow Ft_{10} = T_{1700} + T_{2089}$.

Furthermore,
$8Ft_6 + 2 = 5930 = 1^2 + 77^2 \Rightarrow Ft_6 = T_0 + T_{38}$,

where $T_0 = 0$ and therefore $Ft_6 = T_{38}$. With this, an open question can be raised as follows:
Open Question 1. Aside from $Ft_6 = T_{38} = 741$, is there any other factoriangular number that is also triangular?

Using the same result that a positive integer can be expressed as a sum of two squares only if it contains no prime factor of the form $4k + 3$ occurring in odd multiplicity, the representation as sum of two squares of factoriangular numbers, for $n \leq 20$, were determined and presented in Table 5.

Table 5. Representation as Sum of Two Squares of Factoriangular Numbers for $n \leq 20$

n	**Ft_n**	**Sum of Two Squares**
1	2	$1^2 + 1^2$
2	5	$1^2 + 2^2$
4	34	$3^2 + 5^2$
9	362925	$195^2 + 570^2$
16	20922789888136	$1041930^2 + 4453894^2$
17	355687428096153	$10699077^2 + 15531168^2$

In addition, the factoriangular numbers, for $n \leq 20$, which can be expressed as sum of two triangular numbers or as sum of two squares are concurrently presented in Table 6.

P-ISSN 2350-7756 | E-ISSN 2350-8442 | www.apjmr.com
Asia Pacific Journal of Multidisciplinary Research, Vol. 3, No. 4, November 2015

Table 6. Representation of Factoriangular Numbers, for $n \leq 20$, as Sum of Two Triangular Numbers or as Sum of Two Squares

n	Ft_n	Sum of Two Triangular Numbers	Sum of Two Squares
1	2	$T_1 + T_1$	$1^2 + 1^2$
2	5	---	$1^2 + 2^2$
3	12	$T_3 + T_3$	---
4	34	$T_3 + T_7$	$3^2 + 5^2$
5	135	$T_5 + T_{15}$	---
6	741	$T_{23} + T_{30}$	---
9	362925	---	$195^2 + 570^2$
10	3628855	$T_{269} + T_{2680}$	---
16	20922789888136	---	$1041930^2 + 4453894^2$
17	355687428096153	---	$10699077^2 + 15531168^2$
19	121645100408832190	$T_{226477620} + T_{438175864}$	---
20	2432902008176640210	$T_{746730680} + T_{2075619740}$	---

Another open question is raised as follows:

Open Question 2. Aside from $Ft_1 = 2$ and $Ft_4 = 34$, is there any other factoriangular number that can be expressed both as sum of two triangular numbers and as sum of two squares?

By Theorem 3.11, all factoriangular numbers are trapezoidal and by Definition 3.3 and Definition 3.5, all trapezoidal numbers are polite. Hence, all factoriangular numbers are polite numbers.

CONCLUSIONS

A factoriangular number is defined as a sum of a factorial and its corresponding triangular number. With this definition and other related concepts, the following theorems had been established:

Theorem 1. For $n \geq 1$, Ft_n is the nth factoriangular number, and T_n is the nth triangular number, only the two pairs: $Ft_1 = 2$ and $T_1 = 1$, and $Ft_3 = 12$ and $T_3 = 6$, satisfy the relation $Ft_n = 2T_n$.

Theorem 2. For $n, x \geq 1$, Ft_n is the nth fatoriangular number and T_x is the xth triangular number, $Ft_n = 2T_x$ if and only if $4Ft_n + 1$ is a square.

Theorem 3. For $n, x \geq 1$, Ft_n is the nth triangular number, and T_i is the ith triangular number, $Ft_n = T_x + T_n$ if and only if $8n! + 1$ is a square.

Theorem 4. For $n, x \geq 1$, Ft_n is the nth factoriangular number and T_i is ith triangular number, $Ft_n = T_x + T_y$ if and only if $8Ft_n + 2$ is a sum of two squares.

The following statements are also assumed to be true:

Conjecture 1. For $n, x \geq 1$ and $n \neq x$, there is no pair of factoriangular number, Ft_n, and triangular number, T_x, that satisfy the relation $Ft_n = 2T_x$.

Conjecture 2. $Ft_1 = 2$ and $Ft_3 = 12$ are the only factoriangular numbers that is twice a triangular number.

Conjecture 3. For $n, x \geq 1$ and $n \neq x$, Ft_n is the nth factoriangular number and T_i is the ith triangular number, the only solution for $Ft_n = T_x + T_n$ is $(n, x) = (5, 15)$.

Further, two open questions are hereby raised:

Open Question 1. Aside from $Ft_6 = T_{38} = 741$, is there any other factoriangular number that is also triangular?

Open Question 2. Aside from $Ft_1 = 2$ and $Ft_4 = 34$, is there any other factoriangular number that can be expressed both as sum of two triangular numbers and as sum of two squares?

Some future studies can be conducted to further explore the characteristics of factoriangular numbers, to prove the conjectures stated here and to answer the open questions raised.

REFERENCES

[1] Mathematical Association of America (2008). *Is Beauty Truth in Mathematical Intuition?*. URL: http://goo.gl/kR7Tul

[2] O'Connor, J. J. & Robertson, E. F. (2013). *Quotations by Augustus De Morgan*. URL: http://goo.gl/w2Mq0U.

[3] Heck, B. (2009). *Number Theory*. http://goo.gl/0d1v8v.

[4] O'Connor, J. J. & Robertson, E. F. (2013). *Quotations by Carl Friedrich Gauss*. URL: http://goo.gl/1mYA1s.

[5] Burton, D. M. (1980). *Elementary Number Theory*. Allyn and Bacon, Inc.

[6] Tattersall, J. J. (1999). *Elementary Number Theory in Nine Chapters*. Cambridge University Press.

[7] Weisstein, E. W. Triangular number. *MathWorld – A Wolfram Web Resource.* http://goo.gl/Q46L0O

[8] Leroux, P. (2007). A simple symmetry generating operads related to rooted planar *m*-nary trees and polygonal numbers. *Journal of Integer Sequences*, 10, Art. 07.4.7.

[9] Kane, B. (2009). On two conjectures about mixed sums of squares and triangular numbers. *Journal of Combinatorics and Number Theory*, 1(1), 75-88.

[10] Farkas, H. M. (2006). Sums of squares and triangular numbers. *Online Journal of Analytic Combinatorics*, Issue 1(1), 11pp.

[11] Sun, Z. W. (2007). Mixed sums of squares and triangular numbers. *Acta Arithmetica*, 127(2), 103-113.

[12] Guo, S., Pan, H. & Sun, Z. W. (2007). Mixed sums of squares and triangular numbers (II). *Integers: Electronic Journal of Combinatorial Number Theory*, 7(156), 5pp.

[13] Oh, B. K. & Sun, Z. W. (2009). Mixed sums of squares and triangular numbers (III). *Journal of Number Theory*, 129(4), 964-969.

[14] Carlson, J. (2004). *Square-Triangular Numbers*. doi:10.1.1.177.9241.

[15] Behera, A., Panda, G. K. (1999). On the square roots of triangular numbers. *The Fibonacci Quarterly*, 37, 98-105.

[16] Ray, P. K. (2009). *Balancing and Cobalancing Numbers*. Ph.D. Thesis, National Institute of Technology, Roukela, India.

[17] Keskin, R.& Karaatli O. (2012). Some new properties of balancing numbers and square triangular numbers. *Journal of Integer Sequences*, 15, Art. 12.1.4.

[18] Weisstein, E. W. Factorial. *MathWorld – A Wolfram Web Resource.URL: http://goo.gl/QQxYb*.

[19] Alpern, D. *Factorization using Elliptic Curve Method*. http://www.alpertron.com.ar/ECM.HTM

[20] Dacar, F. (2012). *Fermat's Two Square Theorem for Rationals*. URL: http://goo.gl/wteJeo

Copyrights